Probability

Warm up problem. Alice and Bob each pick, at random, a real number between 0 and 10. Call Alice's number A and Bob's number B. The is the probability that $A \leq B$? What is the probability that $A + B \leq 7$? What is the probability that $A - B \geq 7$?

As illustrated in the warm-up problem, some probability problems can be solved by drawing a picture; this approach is sometimes called *geometric probability.* Other approaches can include experimentation, looking at smaller cases, looking at extreme cases, recursion, or carefully listing possibilities.

1. Alice and Bob play a game with two dice, but they do not use the numbers. Some of the faces are painted red and the others blue. Each player throws the dice in turn. Alice wins when the two top faces are the same color. Bob wins when the colors are different. Each has a 50% chance of winning.. The first die has 5 red faces and 1 blue face. How many red and how many blue are there on the second die?

2. On a plate are 8 strands of cooked spaghetti in a tangled pile. You reach in and grab two random ends and paste them together. You then reach in and grab two more ends and paste them together. You continue this until there are no ends left. What is the probability that when done you have just one loop of spaghetti?

3. There are 6 strings clustered together. One end of each string is at point B (the top), and the other at point A (the bottom). First, two of the ends at point A (randomly) are tied together. Then the two more ends at A are tied together, and then the last two. Next, two ends at point B are tied together. Then the two more at B are tied together, and then the last two. What is the possibility that all the strings will be tied together in one large loop?

4. *The Monte Hall Problem.* In the 60's and 70's a popular TV game show was "Let's Make a Deal," hosted by Monte Hall. The high point was the "Big Deal" in which someone who had earlier won a good prize on the show is invited to gamble for a better prize. For the Big Deal, there were three big doors on the stage; behind one of the doors was a fantastic prize, and behind the other two were things people probably did not want (like a goat or a picture of Henry Kissinger.) Once the participant picked a door (number 1 or 2 or 3), Monte Hall then showed what was behind one of the *other* two doors, always showing one of the bad prizes. He then asked the participant whether she/he would like to switch their choice to the other unselected door. Should the deal maker stay with the original choice or switch or does it not matter? To get an idea, team up with someone else. Use three of the styrofoam cups (doors) and a die (under one cup) as the prize. Try several games with each strategy. Which seems to be better?

5. Suppose you are on a game show. The host puts two envelopes, A and B, in front of you, telling you that one contains twice as much as the other. You select envelope A, reach inside, and there is \$200.00. Now, the host offers you a chance to trade your \$200.00 for the other envelope. Should you?

6. Bill and Yolanda want to meet at in the park . They each agree to arrive between 8:00 PM and 8:30 PM. They also agree that after arrival, a person will wait 12 minutes, then leave if the other has not shown by then. If each of Bill and Yolanda arrive at a random time between 8:00 and 8:30, what is the probability that they will meet?

7. Mark had a ruler that was exactly 36 inches long. His second cousin, Marta, was practicing with her Samurai sword and made two straight slashes at arbitrary spots on the ruler, cutting it into three pieces. What is the probability that the pieces of Mark's ruler can form a triangle?

8. You have three six-sided dice (one red, one green, and one blue.) The numbers of the dice are

   red: $\{1, 4, 4, 4, 4, 4\}$, green: $\{2, 2, 2, 5, 5, 5\}$, blue; $\{3, 3, 3, 3, 3, 6\}$.

   For each pair of dice, which of the pair is more likely to roll the higher number? For each pair, what is the probability that a given die rolls the higher number? (You can make these dice with the materials provided and try some experiments.)

   Now Alice and Bob play a game. Alice chooses a die, then Bob chooses one of the remaining dice. They each roll their die and the higher number wins. Who has the advantage in this game?

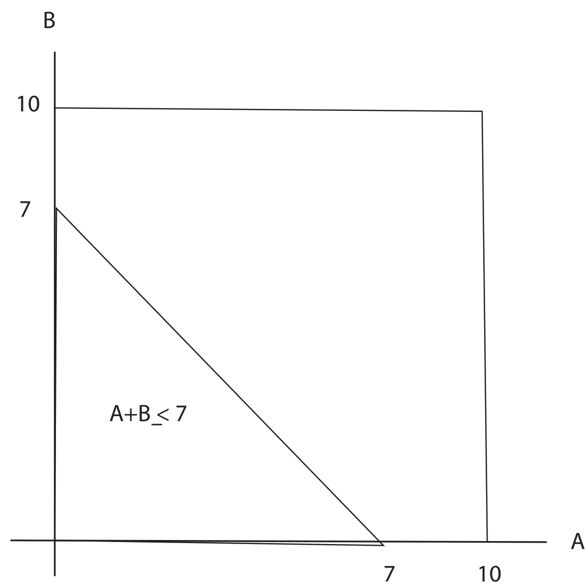
9. Repeat Problem 8 in the case where a player throws two dice of the same color. Which pair wins more often in each case? How does the Alice/Bob game change if each player chooses a set of two dice of the same color?

10. You are a prisoner sentenced to death. However the emperor gives you one chance to live by "winning" a game. You a given two identical bowls, 100 white marbles, and 100 black marbles. You are told that you can divide the marbles and put them in the bowls in any way you want. You will then be blindfolded, and the bowls will be moved. While blindfolded you first select a bowl, then you select a marble from that bowl. If the marble is white, you are set free and live. If the marble is black, you are put to death. How to you set things up to give yourself the best chance of living?

Probability—Notes

Warm-up problem. One nice way to investigate problems of this sort is by graphing. Label the axes in the Cartesian plane A and B. Then a choice of the numbers can be interpreted as a point (A, B) picked randomly (say by throwing a dart) from the 10×10 rectangle with corners (0, 0), (10, 0), (10, 10), (0, 10). The desired probability can then be calculated by figuring out which region of the rectangle contains the points satisfying the given condition. For example, the points with $A+B \leq 7$ are the points triangular region shown below. This region has area $\frac{49}{2}$. Thus the probability that $A+B \leq 7$ is

$$\frac{\text{area of shaded region}}{\text{area of rectangle}} = \frac{\frac{49}{2}}{100} = \frac{49}{200}.$$



Note: Two of the groups "interpreted" real number as integer. This led to some interesting discussions about how the answer can depend on the underlying sample space. In addition, there was discussion about using integers and half-integers as the sample space, and thinking about how the continuous problem can be thought of as a limit of the discrete problems.

1. This can be done by trial and educated guessing; there are only a few ways to assign colors to faces of the dice. For another approach, let r and b be, respectively, the number of red and blue faces on the second die. Now the number or ways the two colors can match is $5r + 1b$, and this must be 18, which is half of the 36 possible outcomes. Thus we have

$$5r + b = 18 \qquad \text{and} \qquad r + b = 6.$$

Hence $r = b = 3$.

2. Recursion is the key here. If we glue together two ends and do not make a loop, then we have the three strand problem. Let $P(n)$ be the probability of ending with one big loop when starting with n strands. Then

$$P(n) = \frac{n-1}{n} P(n-1).$$

This allows us to reduce the problem to a smaller one.

3. How can this problem be changed to one similar to the previous problem?

4. This is a very famous problem and it has received wide coverage in mathematics journals and the public press. Google "Monte Hall Problem" to find lots of discussion about this one. By the way, you are better off switching.

5. This problem too has appeared often in the literature. On the one hand, you have a 50% chance of selecting the envelope with the bigger sum the first time. You should not be able to do better than this. On the other hand, consider the following expected value calculation: the probability is $\frac{1}{2}$ that you have picked the bigger amount and the other envelope has \$100.00. Thus if you switch you gain -100. On the other hand, there is also a 50% chance that you picked the smaller sum and the other envelope has \$400.00. So if you switch and select this one you gain 200. Thus it would seem that be switching your expected return is

$$\frac{1}{2}(-100) + \frac{1}{2}(200) = 50,$$

so you should switch. What is going on here? You can see a lot more discussion by googling "two envelope problem."

6. A nice problem for a geometric probability approach. Label the axes in the Cartesian plane as B and Y. Let B be the number of minutes after 8 that Bill arrives and Y the number of minutes after 8 that Yolanda arrives. Then (B, Y) is a point in a (30, 30) rectangle. What region in this rectangle represents Bill and Yolanda meeting?

7. This too is a very well known and widely discussed problem. It has a very nice geometric solution, but it is a bit tricky. The region to investigate is an equilateral triangle. We need the following fact: Let P be a point inside of equilateral triangle T with altitude h. Drop perpendiculars from P to each side of T and let the lengths of these perpendiculars be x, y, z. Then $x + y + z = h$ (yes, this sum is independent of the location of P!). Now let h be the length of the ruler and x, y, z the lengths of the three pieces. For which points in T are x, y, z the sides lengths for a triangle?

8. This is an example of a set of non-transitive dice. No matter which die you pick first, I can pick one of the other two and be assured that I will roll the higher number with probability greater than $\frac{1}{2}$.

9. With sets of two dice of the same color the sets are still transitive, but the domination goes the other way!

10. This is a good example of the value of investigating extreme cases. You can get the probability of survival close to $\frac{3}{4}$ with the right arrangement.