MTCircular

Summer/Autumn 2017

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A Problem Fit for a Princess Apollonian Gaskets

Polygons and Prejudice Exploring Social Issues

Daydreams in Music Patterns in Scales

Problem Posing Empowering Participants

Math Teachers' Circle π

A NOTE FROM AIM

#playwithmath

Dear Math Teachers' Circle Network,

Summer is exciting for us, because MTC immersion workshops are happening all over the country. We like seeing the updates in real time, on Twitter. Your enthusiasm for all things math and problem solving is contagious!

Here are some recent tweets we enjoyed from MTC immersion workshops in Cleveland, OH; Greeley, CO; and San Jose, CA, respectively:

> *What happens when you cooperate in Blokus? Try and create designs with rotational symmetry.* #toocool #jointhemath – @CrookedRiverMTC
> —
> *Have MnMs, have combinatorial games* @NoCOMTC – @PaulAZeitz
> —
> *Patterns in math are powerful! Always trying to get my Ss to look for #patterns. Stumbled upon Euler's Formula today*. @BayAreaMTCs - @valeriehu6

With each tweet, the underlying message is clear: "Math is social, creative, and energizing." We like that Math Teachers' Circles play a part in encouraging this attitude. We also like seeing teachers treated to a mathematical vacation for a few days. They earned it!

In this issue of the MTCircular, we hope you find some fun interdisciplinary math problems to try with your MTCs. In "A Problem Fit for a Princess," Chris Goff traces the 2000-year history of a fractal that inspired his MTC's logo. In "Polygons and Prejudice," Anne Ho and Tara Craig use a mathematical frame to guide a conversation about social issues. In "Daydreams in Music," Jeremy Aikin and Cory Johnson share a math session motivated by patterns in musical scales. And for those of you looking for ways to further engage your MTC participants' mathematical thinking, Chris Bolognese and Mike Steward's "Using Problem Posing to Empower MTC Participants" will provide plenty of food for thought.

Helping regions and states build networks of MTCs continues to be our biggest priority nationally. We are delighted to help encourage the development of additional regional networks of MTCs by providing consulting, expert mentors, and seed funding when available. Please contact circles@aimath.org if you are interested in finding out more.

Happy problem solving!

Brianna Donaldson

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On the Cover

The cover image depicts a cross-section of a nautilus shell, which grows in a logarithmic spiral. If a ray is drawn from the center, the distance to each intersection point with the spiral will be a constant multiple of the previous distance. For this shell, the ratio is approximately 3.

A Problem Fit for a Princess

Apollonian Gaskets in History

by Christopher Goff

When we first formed the San Joaquin Math Teachers' Circle, we decided to design a logo that prominently featured circles. We ended up basing our logo on an Apollonian gasket, a fractal generated from circles.

Little did we know that this logo would take us on a journey, starting in ancient Greece, passing through seventeenth century Bohemia, moving through twentieth century fractals, and ultimately forming the focus of one of our problem-solving sessions.

Warm-Up: Mutually Tangent Circles

I like to start this session with a warm-up problem about mutually tangent circles (circles that touch at exactly one point):

Find a circle that is tangent to all three circles in Figure 1 below. How many such circles can you find?

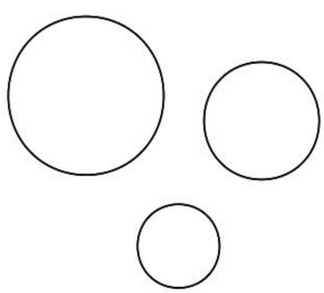


Figure 1. The three given circles.

Teachers usually find a small circle between all three given circles. They also find a big circle that circumscribes all three. In Figure 2 to the right, those two new circles are drawn in red.

It's usually harder to find the three circles in green, as well as the three big blue circles that "contain" two circles each.

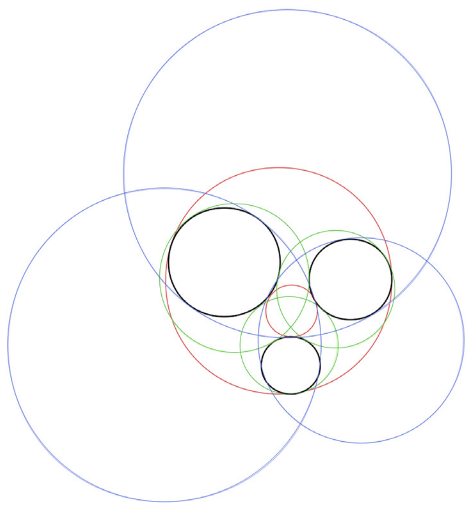


Figure 2. The eight circles that are tangent to all three given circles.

In total, there are eight circles tangent to all three given circles. Through discussion, we distinguish two types of circles: circles that are externally tangent to each other (i.e., the centers of the two tangent circles lie on opposite sides of the mutual tangent line at the point of tangency) or internally tangent (the centers lie on the same side of this line). Another way to see this is to look at whether the circles are curving in opposite directions when they touch (externally tangent) or in the same direction (internally tangent). These concepts will play a role later.

Now, we can classify the eight circles we found based on whether they are externally or internally tangent to the three given circles.

It turns out that our warm-up problem is over 2,000 years old. It was originally posed by Apollonius of Perga, a Greek mathematician who lived in the 200s BCE and was a contemporary of Archimedes. Apollonius wrote a book, *Tangencies,* in which he supposedly stated and solved problems about different geometric objects that were tangent to each other, including our warm-up problem. I say "supposedly" because, unfortunately, the original work is lost. We know about the book from other sources, such as Pappus of Alexandria, who was born over 500 years after Apollonius, and who mentioned *Tangencies* in his own work.

The Logo: Notice and Wonder

At this point, I ask the teachers to take a minute to look at the logo for the San Joaquin MTC. Then, together, we discuss the things we notice and wonder about the logo. This usually brings up some very good questions.

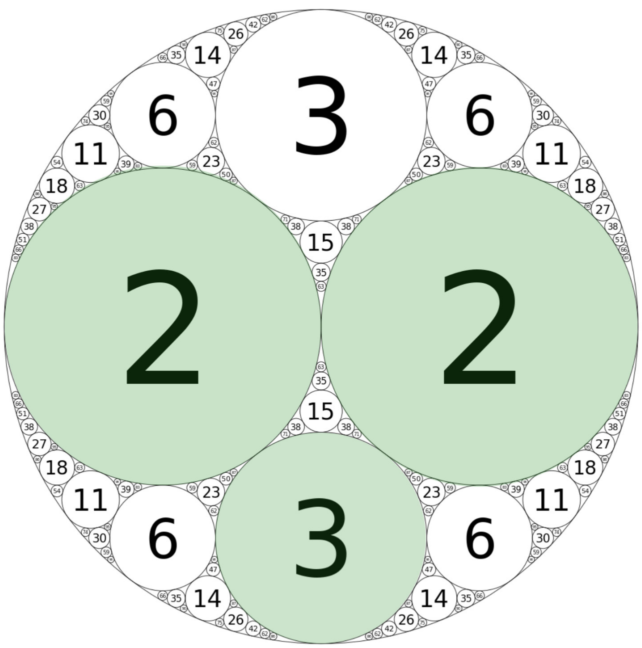


Figure 3. The Apollonian gasket featured in the San Joaquin MTC's logo, with the initial 3 circles shaded in green.

Usually, teachers wonder: what do the numbers mean? They notice that the larger circles contain smaller numbers. Occasionally, they notice a pattern in the numbers along one "arm": 2, 3, 6, 11, 18, 27, 38, ….

Are there other patterns here? Will the numbers in the circles always be integers?

A Princess's Question

Before we attempt to answer all of these questions, let's take another historical detour.

Princess Elisabeth of Bohemia (1618-1680) corresponded with many great intellectuals of her time, and her letters with René Descartes (1596-1650) are especially rich. The two often discussed his philosophical ideas in addition to mathematics; he even dedicated his book *Principia Philosophiae* to her. Perhaps to test the capabilities of his analytic geometric techniques, which were relatively new at the time, Princess Elisabeth asked Descartes to solve the following problem:

Given three circles, find the circle that is externally tangent to all three.

In other words, find the center and the radius of the circle that is "between" the three given circles.

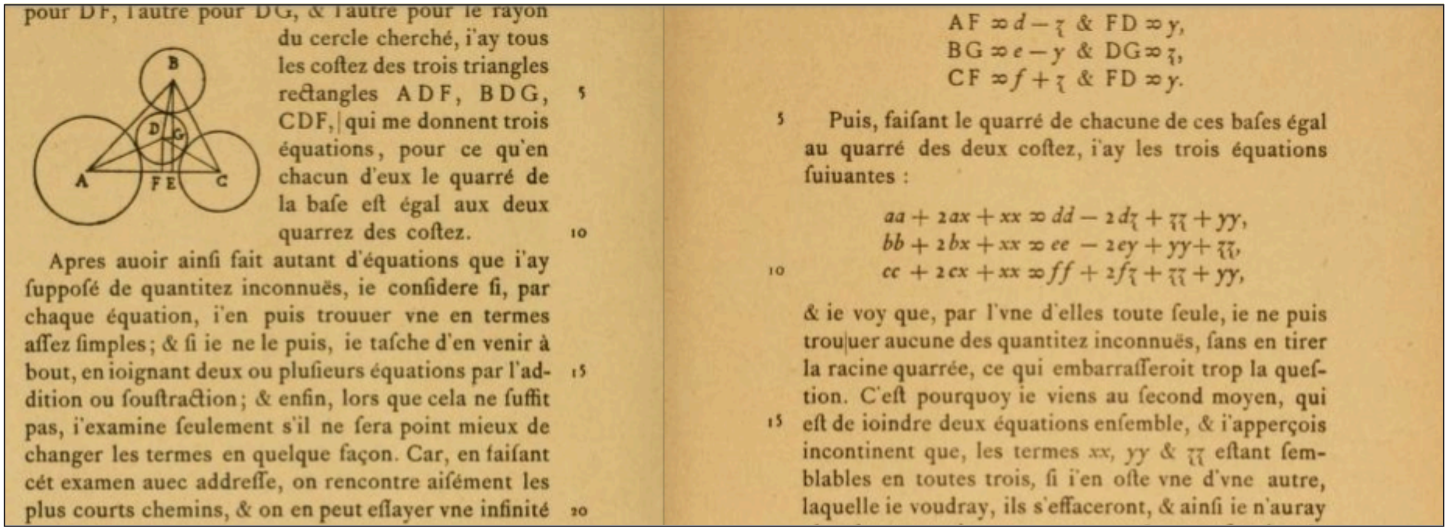
At this point, we take time in our Math Teachers' Circle to try to solve the problem posed by Princess Elisabeth, at least in the case illustrated by our logo, where the three shaded circles are mutually tangent.

After discussing some strategies and partial solutions, we take a look at Descartes' solution. It is unlikely that someone will find as nice a solution as the one Descartes found. I can say that because I had great difficulty working out the algebra involved, even when trying to follow along with Descartes! However, it turns out that when the original three circles are mutually tangent to each other, the solution can be written relatively simply, especially if we utilize the concept of curvature.

A Formula of Curvatures

The curvature k of a circle is defined as the reciprocal of its radius r. That is, $k = 1/r$. Using this notation, if there are four mutually tangent circles with curvatures k_1, k_2, k_3, and k_4, then the curvatures satisfy the following equation:

$$2(k_1^2 + k_2^2 + k_3^2 + k_4^2) = (k_1 + k_2 + k_3 + k_4)^2$$



pour DF, l'autre pour DG, & l'autre pour le rayon du cercle cherché, i'ay tous les coſtez des trois triangles rectangles ADF, BDG, CDF, qui me donnent trois équations, pour ce qu'en chacun d'eux le quarré de la baſe eſt égal aux deux quarrez des coſtez.

Apres auoir ainſi fait autant d'équations que i'ay ſuppoſé de quantitez inconnuës, ie conſidere ſi, par chaque équation, i'en puis trouuer vne en termes aſſez ſimples; & ſi ie ne le puis, ie taſche d'en venir à bout, en ioignant deux ou pluſieurs équations par l'addition ou ſouſtraction; & enfin, lors que cela ne ſuffit pas, i'examine ſeulement s'il ne ſera point mieux de changer les termes en quelque façon. Car, en faiſant cét examen auec addreſſe, on rencontre aiſément les plus courts chemins, & on en peut eſſayer vne infinité

AF ∞ d − z & FD ∞ y,
BG ∞ e − y & DG ∞ z,
CF ∞ f + z & FD ∞ y.

Puis, faiſant le quarré de chacune de ces baſes égal au quarré des deux coſtez, i'ay les trois équations ſuiuantes :

aa + 2ax + xx ∞ dd − 2dz + zz + yy,
bb + 2bx + xx ∞ ee − 2ey + yy + zz,
cc + 2cx + xx ∞ ff + 2fz + zz + yy,

& ie voy que, par l'vne d'elles toute ſeule, ie ne puis trouuer aucune des quantitez inconnuës, ſans en tirer la racine quarrée, ce qui embarraſſeroit trop la queſtion. C'eſt pourquoy ie viens au ſecond moyen, qui eſt de ioindre deux équations enſemble, & i'apperçois incontinent que, les termes xx, yy & zz eſtant ſemblables en toutes trois, ſi i'en oſte vne d'vne autre, laquelle ie voudray, ils s'effaceront, & ainſi ie n'auray

A problem posed by a Princess. From Oeuvres de Descartes, Vol. IV, *published posthumously in 1901.*

If we expand our definition of curvature to include positive and negative values, then we can extend the formula to apply to situations when circles are internally tangent as well as externally tangent. We just need to include a relative minus sign between the curvatures of internally tangent circles. We can even apply the formula if one of the circles has an infinite radius – that is, if it's really a straight line with a curvature of 0.

Given the information that our logo lives inside a large circle of radius 1, at this point, someone usually notices that the numbers inside the circles refer to their curvatures. If we look at the original three circles, which have curvatures 2, 2, and 3, then we can find the curvature k of a circle that is tangent to these three mutually tangent circles.

$$2(2^2 + 2^2 + 3^2 + k^2) = (2 + 2 + 3 + k)^2$$
$$2(17 + k^2) = (7 + k)^2$$
$$k^2 - 14k - 15 = 0$$
$$(k - 15)(k + 1) = 0$$

And so the curvature of the circle we are looking for is either 15 or −1. If you look at the logo, you can see that the circle of curvature 15 is nestled between circles of curvature 2, 2, and 3. You can also see that the big circle has curvature 1, and is internally tangent to these three circles. Hence its curvature is notated as −1.

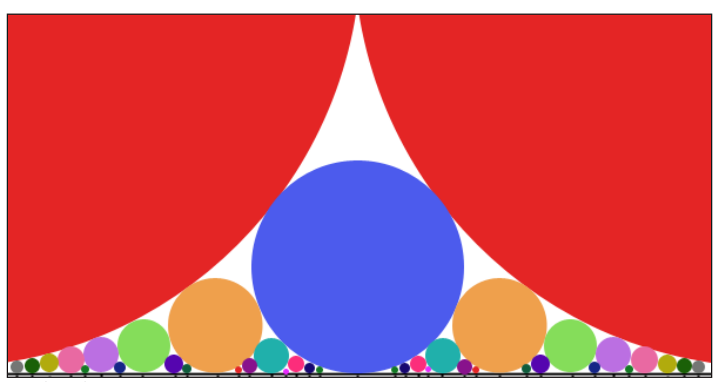
The teachers can now verify many of the curvatures indicated in the San Joaquin MTC logo. They can even create their own similar figure starting with different curvatures. However, given the quadratic nature of the governing formula, one will not always obtain nice integer curvatures. As a result, I usually offer a few "nice" starting options for teachers. Many can be found online.

Extensions

This kind of a shape is a fractal pattern called an Apollonian Gasket, honoring Apollonius and his *Tangencies*. In addition to fractals, other related concepts include Soddy circles and Ford circles. Prominent number theorists have studied some of the properties of the numbers that appear as curvatures in a given gasket pattern. Some have even generalized this formula to higher dimensions.

Our logo took us on quite a journey!

Christopher Goff, a co-founder of the San Joaquin County MTC, is a Professor of Mathematics at the University of the Pacific.



Ford circles.

What's in a Logo?

Many other circles besides the San Joaquin County MTC have designed logos to reflect their mathematical and regional identities. Here is a sampling of logos from MTCs across the country.

Coastal Carolina MTC

Our logo consists of MTC pieced together with CCU, in Coastal Carolina University's colors of teal and bronze. The design in the bottom left corner is a visual representation of the geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$ and sums to 1.

MTC of Austin

We think of our MTC as a bridge between mathematicians and teachers. By inscribing increasingly larger circles inside tangent semicircles, we created a logo that looks like a bridge. The increasingly larger circles also represent our impact over time, as problem solving spreads from teachers to generations of students.

Navajo Nation MTC

The logo uses a Navajo wedding basket as a background. Over the top, it superimposes elements of the seal of the Navajo Nation, which shows the four sacred mountains at the cardinal direction points. We want to hear our participant's pride in saying, "I am Diné and I love mathematics."

North Louisiana MTC

Our logo represents the things we love, Louisiana and math! One of our members sketched the logo using an infinity symbol and the pi symbol to create a fleur-de-lis.

San Diego MTC

Our logo reflects our coastal location. The spiral was generated by dividing a disc into 12 equal parts and dropping perpendiculars from one ray to the adjacent one. This beautiful seashell-like shape gives rise to many interesting math questions of varying depth and difficulty.

South East Ohio MTC

The spiral conveys the sense of an open, inviting community that we hoped to achieve with the SEOMTC. The shape of Ohio in the O gives an immediate sense of place.

Polygons and Prejudice
Introducing Social Issues Through Math
by Anne M. Ho and Tara T. Craig

There is a population of Polygons consisting of Triangles and Squares. They live in a Polygon neighborhood.

Polygons like having neighbors who look like them. Therefore, the Squares are happy when they have mostly Square neighbors, and the Triangles are happy living in majority-Triangle neighborhoods.

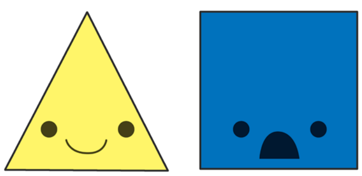


Figure 1. A happy Triangle and an unhappy Square.

This scenario is an adaptation of the online simulation "Parable of the Polygons" (Vi Hart and Nicky Case, 2014), which was based on the paper "Dynamic Models of Segregation" by game theorist Thomas Schelling (1971). Schelling's paper is a study of how small individual biases can lead to collective bias.

We played a board game version of "Parable of the Polygons" with the Coastal Carolina University Math Teachers' Circle during one of our sessions this year. At the start of the session, we broke teachers into small groups, and gave each group a foam board with a preset "neighborhood" filled with happy and unhappy Triangles and Squares. We explained to the participants that the goal of the game is to make all the Polygons happy. There are three rules:

- You can only move Polygons who are unhappy with their immediate neighbors.
- Once they're happy where they are, you can't move them until they're unhappy with their neighbors again.
- All Polygons believe two things: "I want to move if fewer than one third of my neighbors are like me;" and, "I want to move if I have no neighbors."

At first glance, this is simply an exercise in thinking logically and comparing fractions. For example, in Figure 2, the Triangle in the middle has 7 immediate neighbors. Three of these neighbors are Triangles. Since $\frac{3}{7} > \frac{1}{3}$, the middle Triangle is happy.

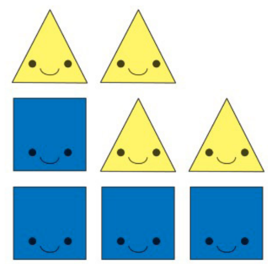


Figure 2. The middle Triangle is happy.

On the other hand, in Figure 3, the middle Square is unhappy because only 1 out of 4 of its neighbors are Squares and $\frac{1}{4} < \frac{1}{3}$.

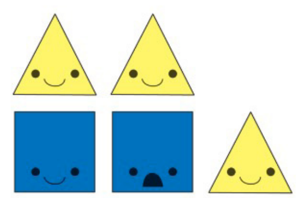


Figure 3. The middle Square is unhappy.

Initially, teachers were focused on winning the game. However, all groups quickly noticed a pattern on their boards, which was that in the process of trying to win, they had segregated the Triangle and Square Polygons.

Suddenly, the game changed. "About halfway through the game I started seeing the pieces as people and not just shapes," said teacher Sydney Logan. "I then realized I didn't want to win anymore if the goal was segregation."

The teachers began to explore solutions and modifications to the game that wouldn't result in segregating the populations. *What if we changed the fraction to be* *larger than or smaller than $\frac{1}{3}$? What happens if we add a third shape, such as Pentagons? How might we modify the rules of the game?*

While playing with modifications, several teachers came up with the idea that an alternating pattern of Triangles and Squares would make all of the Polygons happy. However, with the preset boards and original rules, achieving this pattern was next to impossible. Groups discussed different initial board setups and strategies to make diverse neighborhoods more feasible.

We finished the session with an open discussion. First, we observed how a small individual bias can lead to a larger collective bias, whether the bias is about race, gender, class, occupation, or anything else.

We talked about how the game board would have changed drastically if the Polygons additionally demanded some diversity in their neighborhoods to be happy. Specifically, there would have been more integrated neighborhoods of Triangles and Squares.

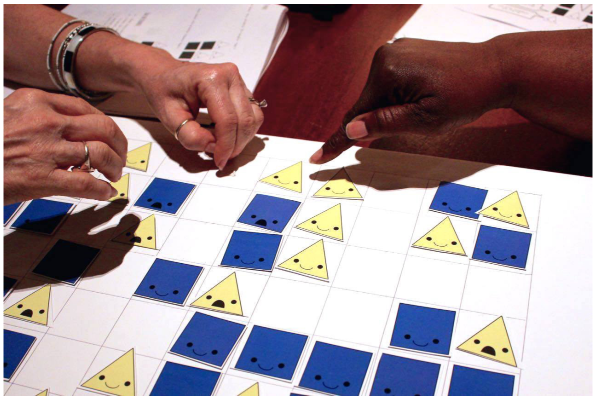
Finally, we talked about the usefulness of mathematical models. Although the simple Polygon population model ignores many socioeconomic reasons for segregation, it still gives a sense of the dynamics in a neighborhood, and it opens the door to having conversations about social issues in a math classroom.

After the session, we asked teachers to play with the original online simulation (http://ncase.me/polygons/), which includes methods to modify the original game rules and graphs of segregation over time when a simulation is run. We also asked everyone to examine their own implicit biases through the research at Project Implicit (https://implicit.harvard.edu/implicit/index.jsp) and to utilize Teaching Tolerance's Social Justice Standards for the classroom (http://www.tolerance.org/social-justice-standards), which are leveled for all the stages in K-12 education and can help with curriculum development.

A math classroom is an uncommon setting in which to have conversations about segregation and bias. However, this game provided a safe and comfortable environment for some of our teachers to talk about these complex issues.

We invite you to do the same.

Anne M. Ho is an assistant professor of mathematics and Tara T. Craig a visiting assistant professor of mathematics, both at Coastal Carolina University. They are co-founders of the Coastal Carolina MTC.



Coastal Carolina MTC participants explore polygon neighborhoods.

Daydreams in Music
Patterns in Musical Scales
by Jeremy Aikin and Cory Johnson

Albert Einstein once said: "If I were not a physicist, I would probably be a musician. I often think in music. I live my daydreams in music. I see my life in terms of music." There is no shortage of examples of mathematicians and scientists who are also musicians. Perhaps it is the abundance of patterns and structure prevalent in music that underpin these common interests. Such patterns can be seen in the very building blocks of music, which motivated the development of an investigation of musical scales for one of our Inland Empire Math Teachers' Circle sessions.

A Model of the Piano

In order to study musical scales through the lens of mathematics, we first developed a mathematical model that will allow us to see and to explore the patterns that arise. While musical scales can be played on many different instruments, the piano may be the most natural instrument to use in our investigation since most people know what it looks like and can easily notice patterns when looking at a piano keyboard.

Figure 1. Periodicity of a piano keyboard.

An observer might notice that the piano keyboard seems periodic, repeating the black keys in groups of two and three. In Figure 1, we have outlined an example of a single period repeated across the keyboard in blue. In music, such a period is called an octave. This periodicity enables us to restrict our attention to a single octave in our study of musical scales.

Figure 2. A single period (or octave).

We asked the participants, "How many keys are there in Figure 2?" The consensus was twelve. Then, we asked, "How did you count the keys?"

Most of the participants had counted the black keys first, and then counted the white keys. Others had counted all the keys in ascending order. This method of counting allowed us to introduce the idea of ascending order and to define the notion of a half-step (**H**) and a whole-step (**W**). A half-step describes the distance between a key on the piano and the neighboring key, in ascending order (for example, from the key labeled 1 to the key labeled 2, or from key 5 to key 6). A whole-step consists of two half-steps (for example, from key 1 to key 3).

We constructed our model for the piano keyboard by bending Figure 2 into a 12-gon so that keys 1 and 12 became neighboring keys. In the resulting model (see Figure 3), adjacent wedges represent one half-step. The numbers that label the wedges in Figure 3 correspond to the numbers that label the piano keys in Figure 2.

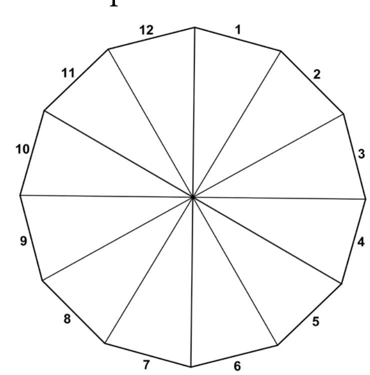


Figure 3. A model of a piano keyboard.

Exploring Musical Scales

In music, a scale is broadly defined as a collection of musical notes arranged in order based on the frequencies of their pitches. Scales are often distinguished by the intervals between these pitches, and the intervals involved in building a scale can vary greatly.

For our purposes, we restricted our definition of a musical scale to include only whole-steps and half-steps. We further assumed that a musical scale begins and ends on the same wedge in our model. In music, scales generally begin and end on the same note, but in different octaves. In this sense, each wedge in our model represents an entire class of equivalent notes.

Based on these assumptions, we arrived at the following mathematical definition of a musical scale:

A musical scale is a shading of wedges in the model so that given any two neighboring wedges, at least one wedge must be shaded.

Note that such a shading produces an associated sequence of **W**'s and **H**'s. For example, a major scale is the sequence **WWHWWWH**. Studying the model, if we begin on the wedge labeled 1, the major scale begins by moving a whole-step to wedge 3, a whole-step to wedge 5, a half-step to wedge 6, a whole-step to wedge 8, a whole-step to wedge 10, a whole-step to wedge 12, and then a half-step back to wedge 1. This amounts to playing the white keys on a piano in ascending order (note that if we instead started on wedge 2 and constructed this scale, there would be a mixture of black and white keys played on the piano).

Once we had established our definition of a musical scale, we asked participants when two musical scales might be equivalent, and when they might be considered different.

After some discussion, participants agreed that two musical scales would be equivalent if they consisted of the exact same collection of shaded wedges on our model. Figure 5 shows the scale given by the sequence **WWWWWW** (in music, this is called a whole-tone scale). Note that if the scale begins on any of the wedges labeled by an odd number, the result is exactly the same collection of wedges. If instead we shade the same sequence on our model beginning on an even labeled wedge, the result is a different collection of wedges. Hence, there seem to be two different types of scales based on this sequence.

Another interesting case is the diminished scale, represented by the sequence **HWHWHWHW** (shown in Figure 6). How many different types of scales exist based on this sequence? Our MTC participants noticed that by studying the rotational symmetries of the model for this scale, it was

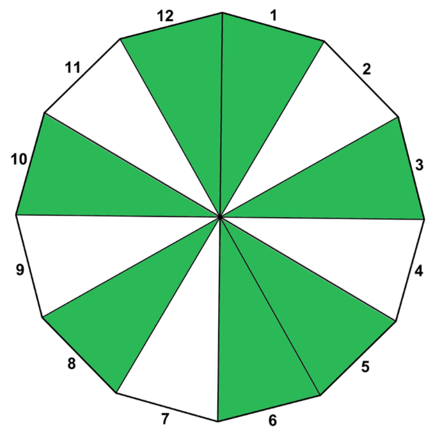


Figure 4. A major scale.

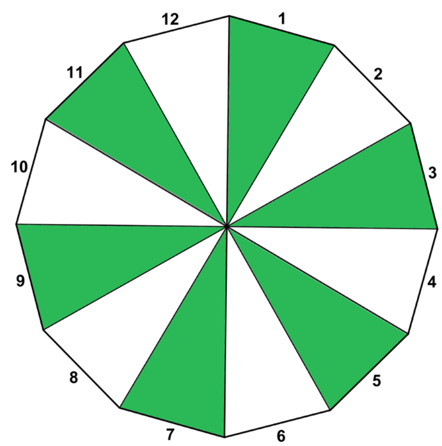


Figure 5. A whole-tone scale.

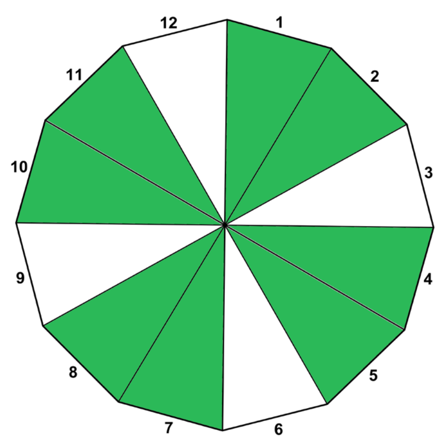


Figure 6. A diminished scale.

possible to visualize the different types of scales having this sequence. For instance, rotating the diagram clockwise by one wedge, or 30°, resulted in a different collection of shaded wedges. However, rotating clockwise by three wedges, or 90°, gave us the same collection of wedges as our initial configuration.

We concluded that there must be three different types of scales having the sequence **HWHWHWHW**. Could the same idea be used to determine the number of different types of major scales (**WWHWWWH**)?

We asked the participants to create their own musical scale by shading a collection of wedges on the model, using only whole and half-steps. This was the most exciting part of this session. From the diagram, they could produce a sequence of **W**'s and **H**'s. By analyzing the rotational symmetries, they were able to determine how many different types of scales could be produced having that sequence.

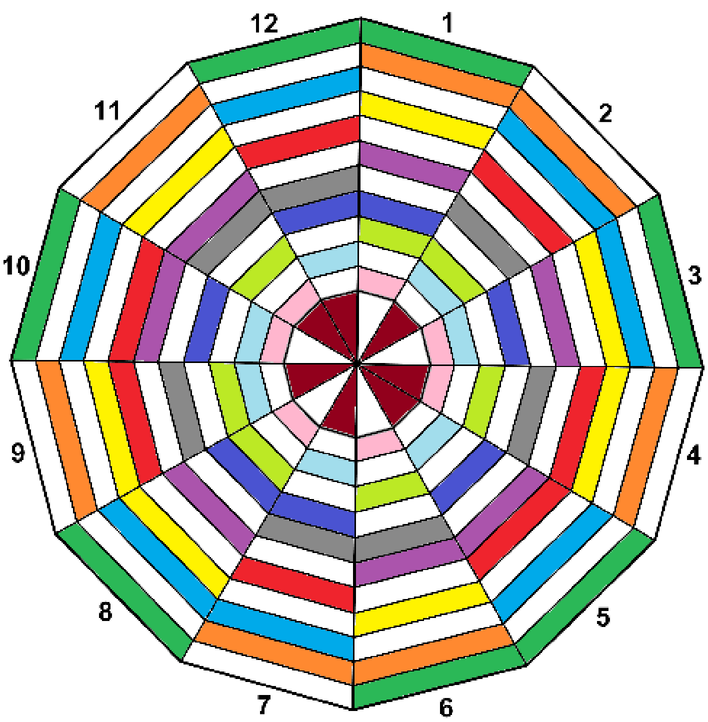


Figure 7. All 12 major scales.

By subdividing the keys in our model, we can keep track of rotational symmetries. This figure shows that there are 12 different major scales.

We asked a natural follow-up question: Is it possible to create musical scales having exactly four different types? More generally, is it possible to create musical scales in which there are exactly five, six, seven, or eight different types? A challenging combinatorial question is: How many different musical scales can our model produce?

Extensions

A nice way to extend this problem would be to consider altering our definition of a musical scale to include other intervals, such as a "three-halves-step" (**T**). This definition might be stated as follows: *A musical scale is a shading of wedges in the model so that given any three consecutive wedges, at least one wedge must be shaded.* For example, shading the wedges in our model that are numbered 1, 4, 6, 7, 8, 11 yields the sequence **TWHHTW** and results in a scale that in music is commonly referred to as a blues scale. Changing the definition creates new scales and allows one to extend the analysis of rotational symmetries.

Reflection

The exploration of musical scales was an engaging session that was accessible to elementary and secondary teachers with and without a musical background. After constructing the model of the piano keyboard, teachers needed minimal instruction from the facilitators. The questions generated were thought-provoking and left room to expand to a deeper level of thinking. As a bonus, we brought in a portable keyboard and played the scales created by our attendees. This brought the session to life in a wonderful way, and enabled us to hear the product of our mathematical thinking.

Jeremy Aikin and Cory Johnson are both assistant professors of mathematics at California State University, San Bernardino. They are co-leaders of the Inland Empire Math Teachers' Circle, which is a member of the Southern California MTC Network.

Problem Posing
A Framework to Empower Participants
by Chris Bolognese and Mike Steward

"The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill. To raise new questions, new possibilities, to regard old questions from a new angle, requires creative imagination and makes real advance in science."

-Albert Einstein, 1938

What is Problem Posing?

Einstein recognized the fundamental importance of *problem posing*, not only as productive practice, but as a gateway to new understanding. The National Council of Teachers of Mathematics (2000) recommends the inclusion of problem posing in mathematics curricula. But what do we mean by problem posing, exactly?

Silver (1994) writes, "Problem posing refers to both the generation of new problems and the re-formulation of given problems. Thus, posing can occur before, during, or after the solution of a problem" (p. 19). Kilpatrick (1987) further suggests that "problem formulation should be viewed not only as a goal of instruction but also as a means of instruction" (p. 123).

In this way, problem posing is both a learning tool and an instructional tool. As a learning tool, it gives students a way to communicate their mathematical ideas and questions. As an instructional tool, it gives teachers insight into students' thinking, and helps teachers guide classroom inquiry.

MTCs can support teachers in understanding and implementing problem posing. What better entry point into this mathematical activity, than for teachers to practice posing problems together? This experience of "thinking like a student" is an important step for teachers who want their students to develop a practice of problem posing.

The following vignettes illustrate the common pitfalls and opportunities of facilitating an MTC.

A Tale of Two Circles, Part 1

Kimberlee is a sixth grade math teacher attending her first MTC. Jenny, the facilitator, says that they will be exploring the brownie problem tonight.

Jenny gives the brownie problem: "After baking a tray of brownies for your students, you leave it on the counter to cool. During that time, a thief takes a piece. How can you share what's left of the brownie between your two classes?"

Jenny distributes a handout for teachers to work on. The handout says:

Consider the brownie as a rectangle in the plane with a rectangular region removed. We define a "cut" to be a single line segment. Answer the following questions under these definitions:

1. *Prove there exists a cut that partitions the remaining brownie into two equal areas.*
2. *Is there a consistent method to construct this cut regardless of the placement of the portion removed?*

Kimberlee hasn't done proofs since tenth grade, so she is not even sure how to start. She draws a picture of the brownie pan on her paper (shown in Figure 1) and notices that the remaining brownie is three pieces. She wonders if this always happens, and what other situations are possible after the cut?

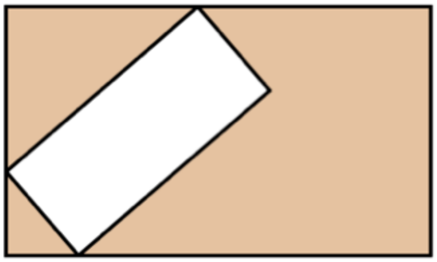


Figure 1. Kimberlee's brownie pan drawing.

Kimberlee calls Jenny over to share her observation. Jenny says, "That's interesting, but it's not what we are thinking about tonight."

A Tale of Two Circles, Part 2

The facilitator, Jermaine, introduces the same brownie problem to his group. "Play around with it," he says. "Pay attention to any questions you have while you explore. Then share your thoughts with your neighbor."

After ten minutes, Jermaine brings the participants back together to share their questions with the whole group. Eitan, a seventh grade math teacher, says, "I'm wondering how many different ways you can cut the brownie so everyone gets an equal share." John, a veteran participant at another table, exclaims, "We were wondering the same thing!"

Jermaine reorganizes the participants into new groups to explore the questions they generated. Eitan and John start to collaborate on their mutual question. For the remainder of the session, Jermaine monitors their progress, occasionally asking tables to share their work with the larger group. When Eitan and John get frustrated with the question they chose to pursue, Jermaine has follow-up questions to help provide insight.

What's the Takeaway?

In the first MTC, Jenny follows a prescribed sequence of questions that may be unnatural or uninteresting to Kimberlee and other participants.

In the second MTC, Jermaine allows Eitan and other participants to follow their own questions, thereby developing a sense of ownership in their pursuits.

How can we make our MTCs look more like the second than the first?

All Questions Are Not Created Equal

The Columbus MTC organically developed a three-level framework for understanding problem posing. The different levels do not pertain to the problem, but rather the solver's relationship to the problem. In this way, problem posing is meant to be a personalized activity. For example, two teachers may rate the same question at different levels, or change their rating as their understanding of a question improves.

In the table below, an example for each level is provided in the context of arithmetic:

Level	Description	Example
Level 1	A question for which you already know the answer.	What is 19 + 4?
Level 2	A question for which you do not know the answer, but have a strategy that you feel will work.	Which is larger, 429 + 384 or 454 + 369?
Level 3	A question for which a strategy for solving is not known to you.	Partition 13 into a sum of one or more positive intergers. Which partition has the largest possible product?

We shared this framework at the 2017 Association of Mathematics Teacher Educators conference. After a brief introduction to the brownie problem, participants generated questions and assigned levels to them:

- How can you cut the remaining brownie in half with a single, one segment cut? (Level 1)
- What happens if the thief removes the entire brownie? (Level 1)
- What if the thief took a piece that isn't the same depth as the rest of the brownie tray? (Level 2)
- Under what conditions will the brownie pieces be mathematically congruent? (Level 2)
- What if the pan is hexagonal and the thief takes a hexagonal piece? (Level 2)
- What is the minimum number of cuts needed to create n pieces of equal area? (Level 2)
- Could you cut the remaining brownie into 3 equal pieces with just two cuts? (Level 3)

Level 2 questions predominated. Participants either did not think of Level 1 or 3 questions, or they were reluctant to share them. As one participant remarked,

> "Level 1 probably isn't even worth saying, because if it's kind of easy, then nobody is going to mess with it.... But distinguishing between Level 3 and Level 2 is important—An idea you want to know, but have no idea where to start; versus an idea that you have a semblance of, an idea that you get. That's a good entry point."

Level 1 questions may seem trivial, but an important part of mathematics is making sense of what is already known. Level 1 questions help us make sense of the current context, and can serve as reference points when exploring a bigger question.

Level 2 and Level 3 questions both expand the boundaries of the context of the problem. If one determines a strategy for solving the problem, one can begin to work on the problem. If all known strategies are exhausted, the question becomes a Level 3.

What's Next?

Before facilitating your next MTC, we ask that you consider these two thoughts:

First, thoughtful facilitation requires paying attention to participants' inquiries. When MTC participants ask the questions, we open a problem up to different perspectives and broaden the range of valid approaches to the scenario. As one of our participants noted, "Problem posing humanizes mathematics."

Second, MTCs can support teachers in facilitating problem posing experiences with their students. However, it's not the full solution, as the dynamics of a classroom and an MTC can vary greatly.

We encourage you to use problem posing and this leveling framework to empower the mathematical thinking of your own participants, and to provide a means for teachers to empower their students.

Chris Bolognese is the K-12 math department chair at Columbus Academy, co-founder of the Columbus MTC, and current president of the Central Ohio Council of Teachers of Mathematics. Mike Steward is an assistant professor of mathematics at the United States Military Academy at West Point. He helped facilitate the Columbus MTC as a graduate student at The Ohio State University.

References

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Members of the Columbus MTC pose and solve problems.

MTCs Featured in KQED MindShift

Math Teachers' Circles were recently featured on KQED's MindShift, a blog that explores the future of learning. The article, "How Playing With Math Helps Teachers Better Empathize With Students," by Katrina Schwartz, focuses on a common theme among teachers who have participated in Math Teachers' Circles: By placing themselves in the position of learner, teachers are able to identify with their students more. "Unlike other professional development opportunities, the focus of these circles is not on lesson plans or pedagogy," writes Schwartz. "Most of the time is spent working on and discussing a problem that the facilitators bring, with the hope that teachers will rediscover what they love about math and how it feels to be a learner." The article profiles MTC leaders Michelle Manes (MTC Hawai'i), Sara Good (Crooked River MTC), and Heather Danforth (AIM MTC).

MTCs Advocating for Math in ESSA Plans

The National Council of Teachers of Mathematics (NCTM), the Math Teachers' Circle Network, and the Association of State Supervisors of Mathematics (ASSM) have undertaken a collaborative effort to help math education leaders identify promising features of state and district ESSA plans. The Every Student Succeeds Act (ESSA) is the education program that replaced No Child Left Behind and restructured how and where federal money for education is allocated. Every district, school, and teacher will be impacted by their state's ESSA plan. This collaborative effort is intended to help the mathematics education community provide detailed and constructive feedback that promotes the advancement of mathematics education explicitly in state and district ESSA plans. Stay tuned for more details on how you can get involved, and contact Brianna Donaldson (brianna@aimath.org) with questions.

Adams, Ghosh Hajra, Manes Win Awards

Adams

Ghosh Hajra

Manes

Kimberly Adams, an Instructor of Mathematics at The University of Tulsa and a member of the Tulsa MTC, was awarded the Kermit E. Brown Award for Teaching Excellence, the highest recognition of teaching excellence given by the College of Engineering and Natural Sciences. Adams is also a Julia Robinson Math Festival coordinator and a frequent MTC facilitator.

Assistant Professor of Mathematics Sayonita Ghosh Hajra received the Hamline University Community Social Justice Award. According to Hamline News, "As a new faculty member to Hamline, she has stepped in right away to Hamline Elementary by incorporating community engagement into her mathematic classrooms with her students." Ghosh Hajra coordinates the St. Paul Elementary MTC.

Michelle Manes, an Associate Professor of Mathematics at the University of Hawai'i at Mānoa and a co-founder of the Math Teachers' Circle of Hawai'i (MaTCH), was honored with the Board of Regents' Medal for Excellence. The award is a tribute to faculty members who exhibit an extraordinary level of subject level mastery and scholarship, teaching effectiveness, and creativity and personal values that benefit students.

Dispatches

Local Updates from Across the Country

California •

Derek Mulkey's lesson on Farey Fractions, developed during the 2015 **AIM MTC** summer workshop, was published in the January/February issue of the Oregon Mathematics Teacher magazine.

- Contributed by Hana Silverstein

The **San Francisco MTC** thanks Proof School and Desmos for generously hosting its sessions this academic year! With 75 total participants, averaging over two dozen each session, folks seemed to have a great experience. Doing the math, we estimate that with the cost of coffee, bagels, potstickers, and noodles, we can bring cool escape-the-textbook mathematics to over 2000 students.

- Contributed by Paul Zeitz

Colorado •

The **Northern Colorado MTC** has been running concurrent sessions for teachers and students. Sessions for teachers run from 5:30 to 8:00, and student sessions run from 5:30 to 7:00. Both groups work on a similar problem for the first hour in separate classrooms, and then, over dinner together, they share solutions and strategies. We believe this structure helps teachers see what students are capable of in terms of problem solving.

- Contributed by Gulden Karakok

Illinois •

The Enterprise Foundation awarded $500 to the **Southwest Chicago MTC** for the 2017-18 school year.

- Contributed by Amanda Harsy

New York •

NYC Community of Adult Math Instructors members E. Appleton, S. Farina, T. Holzer, U. Kotelawala, and M. Trushkowsky published an article in the Spring 2017 issue of the Coalition on Adult Basic Education's Journal of Research and Practice for Adult Literacy, Secondary, and Basic Education. The article, titled "Problem-Posing and Problem-Solving in a Math Teachers' Circle," describes the influence that one of our meetings had on our work with visual patterns and adding problem formulation to the math our students do.

In October 2016, NYC CAMI founding members E. Appleton, S. Farina, T. Holzer, and M. Trushkowsky presented a workshop titled "Teacher-Driven Learning Circles" at the National Council for Teachers of Mathematics (NCTM) Regional Conference in Philadelphia. In April 2017, E. Appleton and M. Trushkowsky presented workshops on mathematical modeling at the National COABE Conference and at the NYC Adult Basic Education Conference.

- Contributed by Mark Trushkowsky

Oklahoma •

The **Tulsa MTC** hosted its fourth annual summer immersion workshop in June 2017 for 37 middle school teachers at the scenic Post Oak Lodge. Nationally recognized math circle leaders Tatiana Shubin and Bob Klein facilitated many sessions. The 22 teachers new to our program gave glowing feedback. Returning teachers thought this summer was the best one yet!

In February 2017, TMTC partnered with Circle Cinema, Tulsa Girls' Math Circle, and The University of Tulsa for a free public screening of the documentary *Navajo Math Circles*. Tatiana Shubin led a Q&A session, and a Native American chef provided food for a reception. The 200-seat theater was packed with math administrators, parents, community education partners, and Native American and Minority Education communities.

Both of these events were made possible through the generous support from NSF, NSA, The University of Tulsa, and local foundations.

- Contributed by Marilyn Howard

Global Math Week

Coming in October!

The Global Math Project aims to engage students and teachers around the world in thinking and talking about the same appealing piece of mathematics during a series of annual Global Math Weeks. Inspired by the work of code.org, which makes coding accessible for millions of students across the globe, we will share the inherent joy, wonder, relevance, and meaning of mathematics with students everywhere and create a forum for the global celebration of creative mathematical thinking.

What Happens During Global Math Week?

The very first Global Math Week takes place this fall. Beginning October 10, 2017, one million students will experience Exploding Dots, a favorite topic for MTCs everywhere that was developed by Global Math Project founding team member James Tanton. During Global Math Week itself, teachers and other math leaders are asked to commit to spending the equivalent of one class period on Exploding Dots, and to share their students' experience with the Global Math Project community through social media. Teachers can choose a low-technology presentation format by using downloadable pdf lesson plans as a guide. Alternatively, they can opt for a technology-intensive experience, developed by the Canadian education technology company Scolab, which will consist of a collection of visually appealing "islands" representing Exploding Dots topics. For those who wish to delve deeper, additional materials will be freely available on the Global Math Project website to support further exploration of place value, arithmetic algorithms, negative numbers, alternative bases, polynomials, formal infinite series, and more.

How Can I Get Involved?

As a Partner Organization for the Global Math Project, the MTC Network is asking our Member Circles to consider becoming involved in some or all of the following ways:

- Encourage your members to register for Global Math Week, and to spread the word to their colleagues and friends.
- Ask members to consider becoming official Global Math Project Ambassadors, who commit to building participation through posting on social media and helping organize local events.
- Host an Exploding Dots training session for Circle members and other interested educators in your area.

See mathematics like you've never seen it before and take part in a global conversation. Get started at http://gmw.globalmathproject.org.

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Comments and suggestions are always welcome at circles@aimath.org.

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