Fooling With Farey Fractions: Improving Understanding of Fractions With Farey Sequences

by Derek Mulkey , Graham Middle School, Mountain View, California

Overview

STUDENTS' PERCEPTIONS OF FRACTIONS are often weighed down by contradictory rules: "Find common denominators." "Don't find common denominators." "Add numerators." "Keep the denominators." "Multiply by the reciprocal of the denominator." The list goes on, making fractions one of the most difficult topics for students of any ages.

This lesson will use Farey fractions to explore the concept of fractions. It will start with an interesting problem: Find a fraction strictly between $^{48}/_{97}$ and $^{49}/_{99}$ with the smallest possible denominator. (Tatiana Shubin) [The specific fractions for the problems may be changed, but this is the concept for the lead-in question.]

Students will work on this problem using different methods. Many will create $^{48.5}/_{98}$, which could be represented as $^{97}/_{196}$. Is this the smallest possible denominator?

Using Farey fractions and creating Farey sequences will allow students to solve the lead-in question and to easily determine where fractions should be placed on a number line. You will move away from this question to an explanation and exploration of Farey sequences.

A Farey sequence F_N is the set of all fractions in lowest terms between 0 and 1 whose denominators do not exceed N, arranged in order of magnitude.

For example, F_6 is $^0/_1$, $^1/_6$, $^1/_5$, $^1/_4$, $^1/_3$, $^2/_5$, $^1/_2$, $^3/_5$, $^2/_3$, $^3/_4$, $^4/_5$, $^5/_6$, $^1/_1$.

These fractions include every reduced fraction between 0 and 1 with a denominator less than or equal to 6. All possible fractions with these constraints are listed and they are in order of magnitude from least to greatest.

For any three successive numbers, the middle number is the mediant of the other two. The term "mediant" applies only to Farey sequences.

To find the mediant of any two numbers in a Farey sequence, you need to add the numerators and add the denominators like this: $^a/_b < {}^{a+c}/_{b+d} < {}^c/_d$.

You should take a moment and verify this process by taking three consecutive values from the above F_6 sequence; let's say $^1/_3$, $^2/_5$ and $^1/_2$. Two-fifths was created from $^1/_3$ and $^1/_2$. You add the numerators $1 + 1 = 2$ and add the denominators, $3 + 2 = 5$ to create the mediant of $^2/_5$. Feel free to try this for any values in the sequence. You can verify that $^2/_5$ is between $^1/_3$ and $^1/_2$ by using equivalent fractions or convert the fractions to decimal form.

This is often how students incorrectly add fractions, but it has a valid use in creating and finding fractions (mediants) between numbers in a Farey sequence and will show students through repetition that the mediant can't be the actual sum of the two fractions because the sum must be greater than both addends.

As students work to create their sequence, they have to attend to precision and check their work. In the discussion they will realize that if they add two fractions, such as $^1/_3$ and $^1/_2$, the sum must be greater that both individual numbers. To create the mediant, students add the numerators and then add the denominators and get a number that is between $^1/_3$ and $^1/_2$ on a number line. This may help students realize, through repetition, that you cannot correctly add fractions by simply adding the numerators and adding the denominators.

After creating different Farey sequences, students can extend to create lists of fractions with a set denominator, such as four: $^1/_4$, $^2/_4$, $^3/_4$. These can be checked against the F_4 Farey sequence to see if all of these values appear in the appropriate Farey sequence. They will appear, but may be in the reduced form. This will allow students to make a connection with equivalent fractions.

NCTM/ Common Core Standards for Mathematical Practice

1. Making sense of problems and persevere in solving them.
3. Construct viable arguments and critique the reasoning of others.
6. Attend to precision.
7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning. Content Standards: CC 6.NS.1 CC 6.NS.6 CC 6.NS.7

Learning objectives

Students will create, organize, and analyze Farey sequences to gain a more complete understanding of fractions and fraction operations.

Materials required

Binder paper, grid paper, pencil, pen, ruler, whiteboard

Instructional Plan

Introduce Problem

Today we will be exploring an interesting area of mathematics, called Farey Fractions, and as the title implies, we will be fooling around with these numbers. Let's start with a definition: A Farey sequence F_N is the set of all fractions in lowest terms between 0 and 1 whose denominators do not exceed N, arranged in order of magnitude.

For example, F_6 is $0/1, 1/6, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 5/6, 1/1$.

These fractions include every reduced fraction between 0 and 1 with a denominator less than or equal to 6. All possible fractions with these constraints are listed and they are in order of magnitude from least to greatest.

You will create Farey sequence by finding the mediants of consecutive fractions with the appropriate denominator. The largest denominator is determined by the N value of the sequence, which means that the largest denominator for the F_6 Farey sequence is 6, which you can see in $1/6$ and $5/6$.

F_1	$0/1, 1/1$
F_2	$0/1, 1/2 \quad , 1/1 \rightarrow 0/1, 1/2, 1/1$
F_3	$0/1, 1/3 \quad , 1/2, 2/3 \quad , 1/1 \rightarrow 0/1, 1/3, 1/2, 2/3, 1/1$
F_4	$0/1, 1/4 \quad , 1/3, 1/2, 2/3, 3/4 \quad , 1/1 \rightarrow 0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1$

Figure 1

Students will be shown how to create the first 3 or 4 Farey sequence. See Figure 1.

From this point students can work independently to create larger Farey sequences.

Whole group checkin #1

Goal # 1: Students should successfully create and organize their Farey sequences to whatever N value they have calculated.

Goal #2: Poll the class and see how far students have gotten. Write down the sequence that students have created on the whiteboard to share with the class.

Goal #3: Now give two prompts to guide thinking and small group discussions: A) What do you notice about this process/patterns/etc? B) What do you wonder about this process/patterns/etc? Take time to write some of these thoughts on the whiteboard, and then tell students it is time for the main challenge.

Prompt

How many fractions are there in the tenth Farey sequence, F_{10}? What is the sum of the fractions in the tenth Farey sequence, F_{10}? Watch for the progress students are making; approximately 10 to 15 minutes.

Work time/student support

Give students time to continue creating Farey sequences and noticing any patterns they find in the new sequence. They should also look back and see if they can find any patterns between the previous sequences.

Whole group check-in #2

Prompt: Can you predict the number of fractions you will have in the F 11 sequence? Can you predict the sum of the F 11 sequence?

Debrief/ wrapUp

Bring the class back together and have students share what they have found.

Questions for Students

Assessment Options: Difference between consecutive fractions. Extensions: Have students calculate the distance (difference) between any two consecutive fractions in a Farey sequence. They can create a formula to find this for any two fractions that are consecutive in some Farey sequence.

If students are interested in rational numbers larger than one, there is a related mathematical sequence, SternBrocot trees, that addresses this and allows students to create sequences that will create improper

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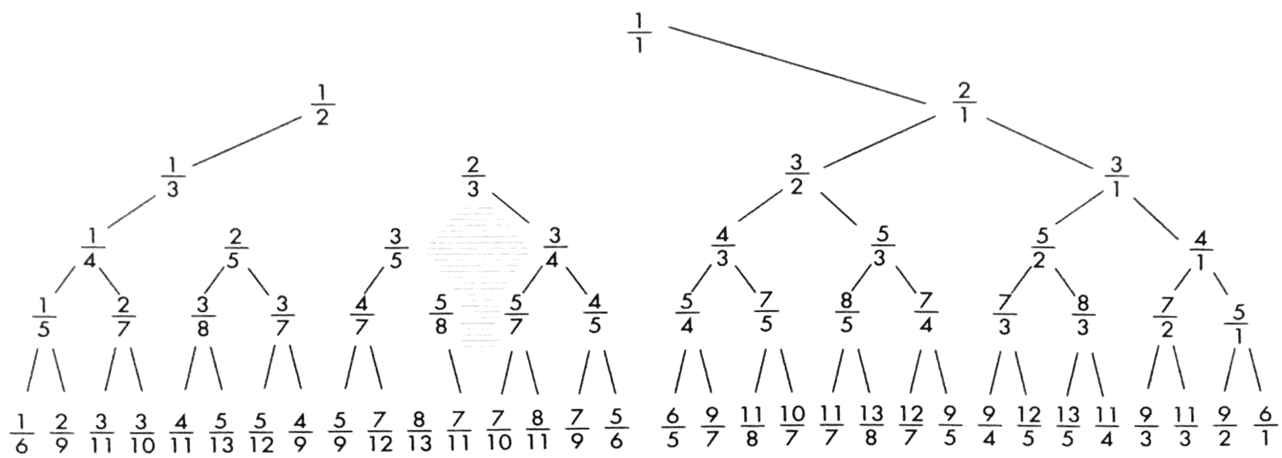
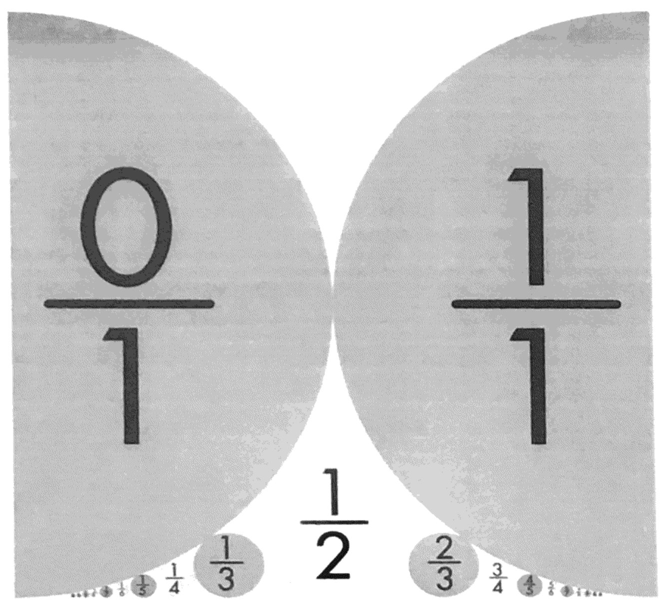
fractions that will contain all rational numbers larger than 1.

There is a graphical representation of the Farey sequences called Ford Circles. Each Farey number in a sequence is represented by a circle tangent to the x-axis at the location of the Farey number. The radius is equal to $^1/_2$(denominator)^2



SternBrocot Tree



Ford Circles

References

Farey Sequences: http://www.cuttheknot. org/blue/Farey.shtml
SternBrocot Trees: http://www.cuttheknot. org/blue/Stern.shtml
Ford Circles: http://www.cuttheknot. org/proofs/fords.shtml
Ford Circles (animation): http://nrich.maths.org/6594