MAGIC SQUARES

KATIE HAYMAKER

Supplies: Paper and pen(cil)

1. Initial setup

Today's topic is *magic squares*. We'll start with two examples. The unique magic square of order one is $\boxed{1}$.

An example of a magic square of order five is:

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

There are a few things to notice about these examples: the magic square of order one is a square 1-by-1 array that contains only the number 1; the magic square of order five is a 5-by-5 square array containing the numbers $1, 2, 3, \ldots, 25$. Each number in the list appears exactly once. Further, if you take the sum of any row ($17 + 24 + 1 + 8 + 15$, for example), or any column ($1+7+13+19+25$, for example), it will always be the same number, called the *magic sum* (in this case, 65). In fact, in order for an array to be a magic square, we also require that the two main diagonals also sum to that same number. This brings us to the general definition.

A (normal) *magic square of order n* is an n-by-n square arrangement of the numbers $\{1, \ldots, n^2\}$ where:

- each number appears exactly once;
- every row, column, and the two main diagonals sum to the same number, the *magic sum*.

The first recorded examples of magic squares showed up over 2000 years ago, and they were long thought to have mystical properties, as their name suggests.

Squares that contain entries from any set of n^2 positive integers, instead of just $\{1, \ldots, n^2\}$ are also known as magic squares. The ones that specifically use the numbers $\{1, \ldots, n^2\}$ are sometimes called *normal magic squares*.

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Unless otherwise stated, we'll be working with normal magic squares (and so we'll drop the "normal" most of the time).

Question 1. Group discussion: Is there a magic square of order 2?

No, there is not. If there were such a square, consider the row and column containing the number 1. Say that the other entry in the row is a and the other column entry is b. The row and column entries add up to the magic sum, so $1+a=1+b$, and thus $a=b$. But the square contains the numbers $\{1,2,3,4\}$, each appearing once.

You can also think of all possible squares with entries 1, 2, 3, and 4, and observe that none of them satisfies the row/column/diagonal sum property. There are 24 squares to consider, but the symmetry of the square allows us to get away with just checking the squares below:

1	2
3	4

1	4
3	2

1	2
4	3

2. Exploration

Question 2. Construct a magic square of order 3.

There are 8 of them (all can be obtained by combinations of rotations and reflections of the first square):

8	1	6
3	5	7
4	9	2

4	3	8
9	5	1
2	7	6

2	9	4
7	5	3
6	1	8

6	7	2
1	5	9
8	3	4

6	1	8
7	5	3
2	9	4

2	7	6
9	5	1
4	3	8

4	9	2
3	5	7
8	1	6

8	3	4
1	5	9
6	7	2

Question 3. For a magic square of order 3 (containing the numbers $\{1,2,\ldots,9\}$), what does the magic sum *have* to be? Try to use a general argument that does not rely on the constructed squares.

The magic sum is 15. The sum of all entries is $1+2+\cdots+9=45$. Since the three rows each add up to the magic sum, the sum must be 45/3=15. This is true for any magic square of order 3.

Question 4. How many distinct magic squares of order 3 are there? When do we consider two squares to be the same?

There is 1, up to rotation and reflections. If you consider a rotation or reflection of the square as distinct, there are 8 (shown above).

To see why the order 3 magic squares above are the *only* order three magic squares, consider the collection of sums with three distinct entries that add up to 15:

$$1 + 5 + 9$$
$$1 + 6 + 8$$
$$2 + 4 + 9$$
$$2 + 5 + 8$$
$$2 + 6 + 7$$
$$3 + 4 + 8$$
$$3 + 5 + 7$$
$$4 + 5 + 6$$

Since 5 is the only number that shows up in four of these sums, it must occupy the middle square. Since 1, 3, 7, and 9 show up in only two sums each, these four numbers must be placed in the four ends of the "+" sign part of the square array. Once the 5 has been placed in the middle square, there are 4 choices for the top of the "+" sign. At that point, there is no longer a choice of how to fill in the bottom of the plus sign. Now consider the lefthand part of the "+" sign. There are two choices remaining for that position. Once that is filled in, the entire square is determined (this is worth trying examples to check). That means there are $4 \times 2 = 8$ possible magic squares of order 3, and they are all rotations and reflections of a single square, as shown in question 2.

Aside: Up to rotation and reflection, there are 880 distinct magic squares of order 4, and 275,305,224 magic squares of order 5. The number of magic squares of order 6 is unknown, but it is estimated to be over 10^{19}.

Question 5. What is the magic sum of a magic square of order n?

Let M be the magic sum and n be the order of the square. The sum of all the entries is

$$1 + 2 + \cdots + n^2 = \frac{(n^2+1)n^2}{2}$$

Since there are n rows, and each row sums to the magic sum, we have

$$nM = \frac{(n^2+1)n^2}{2},$$

so

$$M = \frac{(n^2+1)n}{2}.$$

3. Additional Questions

Groups may wonder – for a magic square of odd order, does the center entry always have to be the median number in the list $\{1, \ldots, n^2\}$? That is, does the center number have to be $\frac{n^2+1}{2}$? The answer is no, and here is an example of an order 5 magic square with 19 as the middle entry. (This example can be constructed using the "knight's move" method decribed in the answer to question 8.)

11	24	7	20	3
17	5	13	21	9
23	6	19	2	15
4	12	25	8	16
10	18	1	14	22

4. Variations on Magic Squares

There are many variations of magic squares, for example: using multiplication instead of addition; magic triangles, cubes, or circles; magic squares within a larger square, similar to Sudoku; concentric magic squares; semi-magic squares that do not satisfy the diagonal sum condition, and many others!

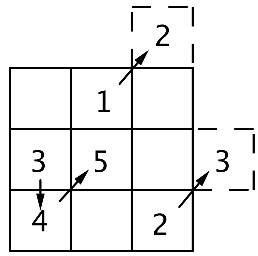
Question 6. Choose one of the variations above, or make up your own. Can you construct one of these magic objects?

5. Extensions: Constructing magic squares of odd order

There are many constructions of magic squares. Here are two constructions for magic squares of odd order.

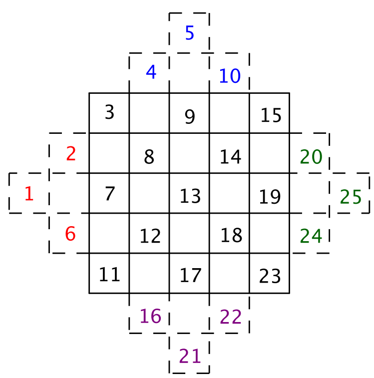
Construction 1

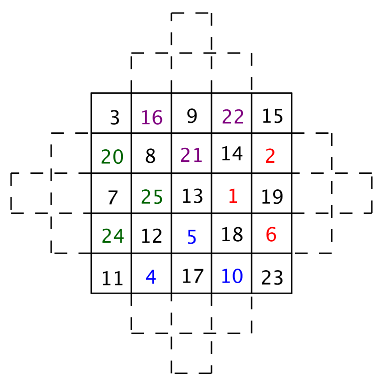
- Start with a 1 in the middle entry of the first row. We will be using "wrap around" in this construction.
- Go up and to the right by one square. With wrap around, we end up one column to the right, in the bottom row. Put a 2 in this square.
- Continue moving up and right and putting the next number, wrapping around when necessary. If you run into a square that is already occupied, just go below the most recently filled in square and place the next number.
- Continue until the magic square is filled.



Construction 2

- Start by drawing triangles of squares around the outside of the magic square.
- Place a 1 in the far left box, and move up and to the right, placing the numbers in increasing order.
- Once you reach the top of the diagonal, go down to the next full diagonal of length n and continue the process until you have placed all of the numbers.
- Thinking of the outer triangular arrays as puzzle pieces, slide them into the place where they fit in the magic square (without changing the orientation of the puzzle piece).





Question 7. Do these methods produce distinct magic squares?

For order 3 they produce the same square since there is only the one. For 5×5 and a 7×7, they produce distinct squares. In general, the second construction is like the first one with a different rule for when you run into an occupied square. In Construction 2, you move two positions to the right of the previously filled in square instead of below it.

The order 5 magic square that results from construction 1 is given below.

Question 8. What other construction methods can you come up for magic squares of odd order? Which methods definitely *do not* work?

One example that would work is the "Knight's move" construction – starting with a 1 in center of the bottom row, and then going up two and right one to place the next number. The move is just like the move of the

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knight in a game of chess. If you come upon a square that is occupied, place the next number directly below the square that was filled in on the previous step.

A method that relies only on vertical or horizontal moves would not work, since the magic sum wouldn't appear as the sum of every row and column.

Question 9. (Difficult) Why do construction methods 1 and 2 always work for odd n?

I do not know of any short solution to this, but it can be done using algebra. Proving that Construction 2 always works seems like it might be easier than Construction 1.

Question 10. Allowing for entries other than $\{1, \ldots, n^2\}$, can you create new magic squares using the ones that we already have?

Multiplying all entries by a constant will always produce another magic square (no longer *normal*). Subtracting every entry from $n^2 + 1$ produces another normal magic square. Shifting rows or columns does not usually work because it tends to destroy the magic sum on the main diagonal. There are various transposition operations that produce new magic squares.

6. Sources

- W. Benson and O. Jacoby, "New recreations with magic squares," Dover Publications, Inc. (1976).
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- E. W. Weisstein, "Magic Square," From *MathWorld–A Wolfram Web Resource*. `http://mathworld.wolfram.com/MagicSquare.html`, accessed February 10, 2014.
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