Patterns in Decimal Expansions

Brian Conrad

September 15, 2016

Types of Decimals

Decimals can be finite or infinite (infinitely many nonzero digits):

$$\frac{3}{8} = .375 = .37500000\ldots, \quad \pi = 3.14159265358979323846 2\ldots.$$

Types of Decimals

Decimals can be finite or infinite (infinitely many nonzero digits):

$$\frac{3}{8} = .375 = .37500000\ldots, \quad \pi = 3.14159265358979323846\ldots.$$

Infinite decimals can be periodic or non-periodic:

$$\frac{1}{7} = .\textcolor{blue}{142857}14285714285\ldots, \qquad \frac{2}{7} = .\textcolor{blue}{285714}28571428571\ldots,$$

$$\sqrt{2} = 1.414213562373095\ldots, \qquad \pi = 3.141592653589793 2\ldots.$$

Types of Decimals

Decimals can be finite or infinite (infinitely many nonzero digits):

$$\frac{3}{8} = .375 = .37500000\ldots, \quad \pi = 3.14159265358979323846 2\ldots.$$

Infinite decimals can be periodic or non-periodic:

$$\frac{1}{7} = .142857142857142 85\ldots, \qquad \frac{2}{7} = .285714285714285 71\ldots,$$

$$\sqrt{2} = 1.414213562373095\ldots, \qquad \pi = 3.141592653589793 2\ldots.$$

Periodic decimals can be purely periodic or eventually periodic:

$$\frac{1}{7} = .142857142857\ldots = .\overline{142857}, \quad \frac{53}{82} = .6\overline{46341}.$$

We will focus on periodic decimals and their patterns.

Theorem (Lambert, 1753)

Consider a real number x.

It is a finite decimal $\Leftrightarrow$ x is a fraction with denominator $2^i 5^j$.

The implication "$\Rightarrow$" is elementary since $10^m = 2^m 5^m$:

$$.38 = \frac{38}{100} = \frac{2 \cdot 19}{2 \cdot 50} = \frac{19}{50} = \frac{19}{2 \cdot 5^2} \quad \text{(reduced form)}$$

Theorem (Lambert, 1753)

Consider a real number x.

It is a finite decimal $\Leftrightarrow$ x is a fraction with denominator $2^i 5^j$.

The implication "$\Rightarrow$" is elementary since $10^m = 2^m 5^m$:

$$.38 = \frac{38}{100} = \frac{2 \cdot 19}{2 \cdot 50} = \frac{19}{50} = \frac{19}{2 \cdot 5^2} \quad \text{(reduced form)}$$

The other implication "$\Leftarrow$" is more interesting:

$$\frac{51}{80} = \frac{51}{2^4 \cdot 5} = \frac{51 \cdot 5^3}{2^4 \cdot 5 \cdot 5^3} = \frac{6375}{10^4} = .6375$$

Fractions are Periodic Decimals: Pure Periodicity, I

$$
\begin{aligned}
x &= .\overline{142857} \text{ (6 digits)} \\
&= .142857142857\ldots
\end{aligned}
$$

Fractions are Periodic Decimals: Pure Periodicity, I

$$\begin{aligned} x &= .\overline{142857} \text{ (6 digits)} \\ &= .142857142857\ldots \\ 10^6 x &= 142857.142857142857\ldots = 142857 + x \end{aligned}$$

Fractions are Periodic Decimals: Pure Periodicity, I

$$\begin{aligned} x &= .\overline{142857} \text{ (6 digits)} \\ &= .142857142857\ldots \\ 10^6 x &= 142857.142857142857\ldots = 142857 + x \\ \text{so } (10^6 - 1)x &= 142857 \end{aligned}$$

Fractions are Periodic Decimals: Pure Periodicity, I

$$\begin{aligned}
x &= .\overline{142857} \text{ (6 digits)} \\
&= .142857142857\ldots \\
10^6 x &= 142857.142857142857\ldots = 142857 + x \\
\text{so } (10^6 - 1)x &= 142857 \\
999999x &= 142857
\end{aligned}$$

Fractions are Periodic Decimals: Pure Periodicity, I

$$\begin{aligned}
x &= .\overline{142857} \text{ (6 digits)} \\
&= .142857142857\ldots \\
10^6 x &= 142857.142857142857\ldots = 142857 + x \\
\text{so } (10^6 - 1)x &= 142857 \\
999999x &= 142857 \\
x &= \frac{142857}{999999}
\end{aligned}$$

Fractions are Periodic Decimals: Pure Periodicity, I

$$\begin{aligned}
x &= .\overline{142857} \text{ (6 digits)} \\
&= .142857142857\ldots \\
10^6 x &= 142857.142857142857\ldots = 142857 + x \\
\text{so } (10^6 - 1)x &= 142857 \\
999999x &= 142857 \\
x &= \frac{142857}{999999}
\end{aligned}$$

But $999999 = 999 \cdot 1001 = (9 \cdot 111) \cdot (11 \cdot 91) = 3^3 \cdot 37 \cdot (11 \cdot 7 \cdot 13)$, and $3^3 \cdot 37 \cdot 11 \cdot 13$ is a factor of 142857, yielding that $x = 1/7$.

$$\begin{aligned} x &= .\overline{142857} \text{ (6 digits)} \\ &= .142857142857\ldots \\ 10^6 x &= 142857.142857142857\ldots = 142857 + x \\ \text{so } (10^6 - 1)x &= 142857 \\ 999999x &= 142857 \\ x &= \frac{142857}{999999} \end{aligned}$$

But $999999 = 999 \cdot 1001 = (9 \cdot 111) \cdot (11 \cdot 91) = 3^3 \cdot 37 \cdot (11 \cdot 7 \cdot 13)$, and $3^3 \cdot 37 \cdot 11 \cdot 13$ is a factor of 142857, yielding that $x = 1/7$.

In general, $.\overline{c_1 c_2 \ldots c_d} = \dfrac{c_1 c_2 \ldots c_d}{10^d - 1} = \dfrac{c_1 c_2 \ldots c_d}{\underbrace{99 \ldots 9}_{d \text{ digits}}}$, so in reduced form the denominator is *not* divisible by 2 or 5 (as $10^d - 1$ is not).

Fractions are Periodic Decimals: Pure Periodicity, I

$$
\begin{aligned}
x &= .\overline{142857} \text{ (6 digits)} \\
&= .142857142857\ldots \\
10^6 x &= 142857.142857142857\ldots = 142857 + x \\
\text{so } (10^6 - 1)x &= 142857 \\
999999x &= 142857 \\
x &= \frac{142857}{999999}
\end{aligned}
$$

But $999999 = 999 \cdot 1001 = (9 \cdot 111) \cdot (11 \cdot 91) = 3^3 \cdot 37 \cdot (11 \cdot 7 \cdot 13)$, and $3^3 \cdot 37 \cdot 11 \cdot 13$ is a factor of 142857, yielding that $x = 1/7$.

In general, $.\overline{c_1 c_2 \ldots c_d} = \frac{c_1 c_2 \ldots c_d}{10^d - 1} = \frac{c_1 c_2 \ldots c_d}{\underbrace{99\ldots9}_{d \text{ digits}}}$, so in reduced form the denominator is *not* divisible by 2 or 5 (as $10^d - 1$ is not).

Let's work on Exercise 1.

Fractions are Periodic Decimals: Pure Periodicity, II

Which reduced a/n with $0 < a < n$ have purely repeating decimal; i.e., $a/n = A/(10^d - 1)$ (says some $nm = 10^d - 1$, $d > 0$)?

Necessary condition: n not divisible by 2 or 5. Is it sufficient?

Fractions are Periodic Decimals: Pure Periodicity, II

Which reduced a/n with $0 < a < n$ have purely repeating decimal; i.e., $a/n = A/(10^d - 1)$ (says some $nm = 10^d - 1$, $d > 0$)?

Necessary condition: n not divisible by 2 or 5. Is it sufficient? Here are examples related to Exer. 1(i):

$n = 7$:

Fractions are Periodic Decimals: Pure Periodicity, II

Which reduced a/n with $0 < a < n$ have purely repeating decimal; i.e., $a/n = A/(10^d - 1)$ (says some $nm = 10^d - 1$, $d > 0$)?

Necessary condition: n not divisible by 2 or 5. Is it sufficient? Here are examples related to Exer. 1(i):

$$n = 7: \quad 7 \cdot 142857 = 999999 \Rightarrow \frac{1}{7} = \frac{142857}{999999} = .\overline{142857}$$

Fractions are Periodic Decimals: Pure Periodicity, II

Which reduced a/n with $0 < a < n$ have purely repeating decimal; i.e., $a/n = A/(10^d - 1)$ (says some $nm = 10^d - 1$, $d > 0$)?

Necessary condition: n not divisible by 2 or 5. Is it sufficient? Here are examples related to Exer. 1(i):

$$n = 7 : \quad 7 \cdot 142857 = 999999 \Rightarrow \frac{1}{7} = \frac{142857}{999999} = .\overline{142857}$$

$n = 27$:

Fractions are Periodic Decimals: Pure Periodicity, II

Which reduced a/n with $0 < a < n$ have purely repeating decimal; i.e., $a/n = A/(10^d - 1)$ (says some $nm = 10^d - 1$, $d > 0$)?

Necessary condition: n not divisible by 2 or 5. Is it sufficient? Here are examples related to Exer. 1(i):

$$n = 7 : \quad 7 \cdot 142857 = 999999 \Rightarrow \frac{1}{7} = \frac{142857}{999999} = .\overline{142857}$$

$$n = 27 : \quad 27 \cdot 37 = 999 \Rightarrow \frac{14}{27} = \frac{14 \cdot 37}{27 \cdot 37} = \frac{518}{999} = .\overline{518}$$

Which reduced a/n with $0 < a < n$ have purely repeating decimal; i.e., $a/n = A/(10^d - 1)$ (says some $nm = 10^d - 1$, $d > 0$)?

Necessary condition: n not divisible by 2 or 5. Is it sufficient? Here are examples related to Exer. 1(i):

$$n = 7 : \quad 7 \cdot 142857 = 999999 \Rightarrow \frac{1}{7} = \frac{142857}{999999} = .\overline{142857}$$

$$n = 27 : \quad 27 \cdot 37 = 999 \Rightarrow \frac{14}{27} = \frac{14 \cdot 37}{27 \cdot 37} = \frac{518}{999} = .\overline{518}$$

$n = 21$:

Fractions are Periodic Decimals: Pure Periodicity, II

Which reduced a/n with $0 < a < n$ have purely repeating decimal; i.e., $a/n = A/(10^d - 1)$ (says some $nm = 10^d - 1$, $d > 0$)?

Necessary condition: n not divisible by 2 or 5. Is it sufficient? Here are examples related to Exer. 1(i):

$$n = 7: \quad 7 \cdot 142857 = 999999 \Rightarrow \frac{1}{7} = \frac{142857}{999999} = .\overline{142857}$$

$$n = 27: \quad 27 \cdot 37 = 999 \Rightarrow \frac{14}{27} = \frac{14 \cdot 37}{27 \cdot 37} = \frac{518}{999} = .\overline{518}$$

$$n = 21: \quad 21 \cdot 47619 = 999999 \Rightarrow \frac{1}{21} = \frac{47619}{21 \cdot 47919} = .\overline{047619}$$

Which reduced a/n with $0 < a < n$ have purely repeating decimal; i.e., $a/n = A/(10^d - 1)$ (says some $nm = 10^d - 1$, $d > 0$)?

Necessary condition: n not divisible by 2 or 5. Is it sufficient? Here are examples related to Exer. 1(i):

$$n = 7: \quad 7 \cdot 142857 = 999999 \Rightarrow \frac{1}{7} = \frac{142857}{999999} = .\overline{142857}$$

$$n = 27: \quad 27 \cdot 37 = 999 \Rightarrow \frac{14}{27} = \frac{14 \cdot 37}{27 \cdot 37} = \frac{518}{999} = .\overline{518}$$

$$n = 21: \quad 21 \cdot 47619 = 999999 \Rightarrow \frac{1}{21} = \frac{47619}{21 \cdot 47919} = .\overline{047619}$$

If $nm = 10^d - 1$ and $0 < a < n$, $\dfrac{a}{n} = \dfrac{am}{nm} = \dfrac{am}{10^d - 1} = \dfrac{am}{\underbrace{99\ldots9}_{d \text{ digits}}}$,

purely periodic since $am < nm = 9\ldots9$.

Which reduced a/n with $0 < a < n$ have purely repeating decimal; i.e., $a/n = A/(10^d - 1)$ (says some $nm = 10^d - 1$, $d > 0$)?

Necessary condition: n not divisible by 2 or 5. Is it sufficient? Here are examples related to Exer. 1(i):

$$n = 7: \quad 7 \cdot 142857 = 999999 \Rightarrow \frac{1}{7} = \frac{142857}{999999} = .\overline{142857}$$

$$n = 27: \quad 27 \cdot 37 = 999 \Rightarrow \frac{14}{27} = \frac{14 \cdot 37}{27 \cdot 37} = \frac{518}{999} = .\overline{518}$$

$$n = 21: \quad 21 \cdot 47619 = 999999 \Rightarrow \frac{1}{21} = \frac{47619}{21 \cdot 47919} = .\overline{047619}$$

If $nm = 10^d - 1$ and $0 < a < n$, $\dfrac{a}{n} = \dfrac{am}{nm} = \dfrac{am}{10^d - 1} = \dfrac{am}{\underbrace{99\ldots9}_{d \text{ digits}}}$, purely periodic since $am < nm = 9\ldots9$.

Theorem

Every n not div. by 2 or 5 has a multiple of form $10^d - 1$ ($d > 0$).

Theorem

Any n not div. by 2 or 5 has a multiple of form $10^d - 1$ $(d > 0)$.

How do we show this is *always* true?

Fractions are Periodic Decimals: Pure Periodicity, II

Theorem

Any n not div. by 2 or 5 has a multiple of form $10^d - 1$ $(d > 0)$.

How do we show this is *always* true?

Idea: Look at remainders when 1, 10, $10^2, 10^3, \dots$ are divided by n.

Theorem

Any n not div. by 2 or 5 has a multiple of form $10^d - 1$ ($d > 0$).

How do we show this is *always* true?
Idea: Look at remainders when 1, 10, $10^2, 10^3, \ldots$ are divided by n.

Example

Try $n = 27$. When 10^k is divided by 27, let r_k be the remainder.
For instance, $10^2 = 100 = 27 \cdot 3 + 19$, so $r_2 = 19$.

k	0	1	2	3	4	5	6
r_k	1	10	19	1	10	19	1

Theorem

Any n not div. by 2 or 5 has a multiple of form $10^d - 1$ $(d > 0)$.

How do we show this is *always* true?
Idea: Look at remainders when 1, 10, $10^2, 10^3, \ldots$ are divided by n.

Example

Try $n = 27$. When 10^k is divided by 27, let r_k be the remainder.
For instance, $10^2 = 100 = 27 \cdot 3 + 19$, so $r_2 = 19$.

k	0	1	2	3	4	5	6
r_k	1	10	19	1	10	19	1

Look for 10^k's where the r_k's are the same. Then subtract!

Fractions are Periodic Decimals: Pure Periodicity, II

Theorem

Any n not div. by 2 or 5 has a multiple of form $10^d - 1$ ($d > 0$).

How do we show this is *always* true?
Idea: Look at remainders when $1, 10, 10^2, 10^3, \ldots$ are divided by n.

Example

Try $n = 27$. When 10^k is divided by 27, let r_k be the remainder.
For instance, $10^2 = 100 = 27 \cdot 3 + 19$, so $r_2 = 19$.

k	0	1	2	3	4	5	6
r_k	1	10	19	1	10	19	1

Look for 10^k's where the r_k's are the same. Then subtract!
$999 = 10^3 - 10^0 = (27 \cdot 37 + 1) - (27 \cdot 0 + 1) = \mathbf{27} \cdot 37.$
$9990 = 10^4 - 10^1 = (27 \cdot 370 + 10) - (27 \cdot 0 + 10) = \mathbf{27} \cdot 370.$

Fractions are Periodic Decimals: Pure Periodicity, II

Theorem

Any n not div. by 2 or 5 has a multiple $10^d - 1$ with $1 \leq d \leq n-1$.

We will use "repeated remainders" to prove this result.

Theorem

Any n not div. by 2 or 5 has a multiple $10^d - 1$ with $1 \leq d \leq n-1$.

We will use "repeated remainders" to prove this result.

Proof.

Only $n-1$ possible remainders (all non-zero!) when dividing by n, so can find *distinct* 10^k and 10^ℓ (say $k > \ell$) with *same* remainder:

$$10^k = nq + r, \; 10^\ell = nq' + r \quad \Rightarrow \quad 10^k - 10^\ell = n(q - q')$$

Fractions are Periodic Decimals: Pure Periodicity, II

Theorem

Any n not div. by 2 or 5 has a multiple $10^d - 1$ with $1 \leq d \leq n-1$.

We will use "repeated remainders" to prove this result.

Proof.

Only $n - 1$ possible remainders (all non-zero!) when dividing by n, so can find *distinct* 10^k and 10^ℓ (say $k > \ell$) with *same* remainder:

$$10^k = nq + r,\ 10^\ell = nq' + r \quad \Rightarrow \quad 10^k - 10^\ell = n(q - q')$$
$$10^\ell(10^{k-\ell} - 1) = n(q - q') \quad \Rightarrow \quad n \text{ is factor of } 10^\ell(10^{k-\ell} - 1).$$

Theorem

Any n not div. by 2 or 5 has a multiple $10^d - 1$ *with* $1 \leq d \leq n - 1$.

We will use "repeated remainders" to prove this result.

Proof.

Only $n - 1$ possible remainders (all non-zero!) when dividing by n, so can find *distinct* 10^k and 10^ℓ (say $k > \ell$) with *same* remainder:

$$10^k = nq + r,\ 10^\ell = nq' + r \quad \Rightarrow \quad 10^k - 10^\ell = n(q - q')$$

$$10^\ell(10^{k-\ell} - 1) = n(q - q') \quad \Rightarrow \quad n \text{ is factor of } 10^\ell(10^{k-\ell} - 1).$$

So n not divisible by 2 or 5 $\Rightarrow$ n is factor of $10^{k-\ell} - 1$.

Theorem

Any n not div. by 2 or 5 has a multiple $10^d - 1$ with $1 \le d \le n-1$.

We will use "repeated remainders" to prove this result.

Proof.

Only $n-1$ possible remainders (all non-zero!) when dividing by n, so can find *distinct* 10^k and 10^ℓ (say $k > \ell$) with *same* remainder:

$$10^k = nq + r,\ 10^\ell = nq' + r \quad \Rightarrow \quad 10^k - 10^\ell = n(q - q')$$

$$10^\ell(10^{k-\ell} - 1) = n(q - q') \quad \Rightarrow \quad n \text{ is factor of } 10^\ell(10^{k-\ell} - 1).$$

$$\text{So } n \text{ not divisible by } 2 \text{ or } 5 \quad \Rightarrow \quad n \text{ is factor of } 10^{k-\ell} - 1.$$

$$\text{Can pick } k \text{ and } \ell \text{ in } 1, \ldots, n \quad \Rightarrow \quad 1 \le k - \ell \le n - 1.$$

□

Theorem

Any n not div. by 2 or 5 has a multiple $10^d - 1$ with $1 \leq d \leq n-1$.

We will use "repeated remainders" to prove this result.

Proof.

Only $n-1$ possible remainders (all non-zero!) when dividing by n, so can find *distinct* 10^k and 10^ℓ (say $k > \ell$) with *same* remainder:

$$
\begin{aligned}
10^k = nq + r,\ 10^\ell = nq' + r &\quad\Rightarrow\quad 10^k - 10^\ell = n(q - q') \\
10^\ell(10^{k-\ell} - 1) = n(q - q') &\quad\Rightarrow\quad n \text{ is factor of } 10^\ell(10^{k-\ell} - 1). \\
\text{So } n \text{ not divisible by } 2 \text{ or } 5 &\quad\Rightarrow\quad n \text{ is factor of } 10^{k-\ell} - 1. \\
\text{Can pick } k \text{ and } \ell \text{ in } 1, \ldots, n &\quad\Rightarrow\quad 1 \leq k - \ell \leq n - 1.
\end{aligned}
$$

□

Let's now apply this Theorem with examples in Exercise 2.

Fractions are Periodic Decimals: Eventual Periodicity

Fractions with a denominator that is $2^i 5^j$ ($i > 0$ or $j > 0$) times something > 1 without factors of 2 or 5 have *eventually* periodic (but not purely periodic) decimals:

Fractions are Periodic Decimals: Eventual Periodicity

Fractions with a denominator that is $2^i 5^j$ ($i > 0$ or $j > 0$) times something > 1 without factors of 2 or 5 have *eventually* periodic (but not purely periodic) decimals:

$$\frac{46}{105} = \frac{46}{5 \cdot 21} = \frac{2 \cdot 46}{2 \cdot 5 \cdot 21} = \frac{1}{10}\frac{92}{21} = \frac{1}{10}\left(4 + \frac{8}{21}\right) = .4{+}?.$$

Here we split $92/21$ into fractional and integer parts (so no "carrying" below, since $0 < 8/21 < 1$; just shift a decimal point).
What is 8/21 as a decimal?

Fractions are Periodic Decimals: Eventual Periodicity

Fractions with a denominator that is $2^i 5^j$ ($i > 0$ or $j > 0$) times something > 1 without factors of 2 or 5 have *eventually* periodic (but not purely periodic) decimals:

$$\frac{46}{105} = \frac{46}{5 \cdot 21} = \frac{2 \cdot 46}{2 \cdot 5 \cdot 21} = \frac{1}{10}\frac{92}{21} = \frac{1}{10}\left(4 + \frac{8}{21}\right) = .4 + ?.$$

Here we split $92/21$ into fractional and integer parts (so no "carrying" below, since $0 < 8/21 < 1$; just shift a decimal point). **What is 8/21 as a decimal?** Using Exercise 1(i):

$$21 \cdot 47619 = 999999 \Rightarrow \frac{8}{21} = \frac{8 \cdot 47619}{999999} = \frac{380952}{999999} = .\overline{380952},$$

Fractions with a denominator that is 2^i5^j ($i > 0$ or $j > 0$) times something > 1 without factors of 2 or 5 have *eventually* periodic (but not purely periodic) decimals:

$$\frac{46}{105} = \frac{46}{5 \cdot 21} = \frac{2 \cdot 46}{2 \cdot 5 \cdot 21} = \frac{1}{10}\frac{92}{21} = \frac{1}{10}\left(4 + \frac{8}{21}\right) = .4+?.$$

Here we split $92/21$ into fractional and integer parts (so no "carrying" below, since $0 < 8/21 < 1$; just shift a decimal point). **What is 8/21 as a decimal?** Using Exercise 1(i):

$$21 \cdot 47619 = 999999 \Rightarrow \frac{8}{21} = \frac{8 \cdot 47619}{999999} = \frac{380952}{999999} = .\overline{380952},$$

so (without carrying!) we recover part of Exercise 1(ii):

$$\frac{46}{105} = .4 + \frac{1}{10}\frac{8}{21} = .4 + \frac{1}{10}(.\overline{380952}) = .4 + .0\overline{380952} = .4\overline{380952}.$$

Fractions are Periodic Decimals: Eventual Periodicity

Fractions with a denominator that is $2^i 5^j$ ($i > 0$ or $j > 0$) times something > 1 without factors of 2 or 5 have *eventually* periodic (but not purely periodic) decimals:

$$\frac{3}{44} = \frac{3}{2^2 \cdot 11} = \frac{5^2 \cdot 3}{5^2 \cdot 2^2 \cdot 11} = \frac{1}{100}\frac{75}{11} = \frac{1}{100}\left(6 + \frac{9}{11}\right) = .06 + ?.$$

(split $\frac{75}{11}$ into integer and fractional parts) **Repeat the method.**

Fractions are Periodic Decimals: Eventual Periodicity

Fractions with a denominator that is $2^i 5^j$ ($i > 0$ or $j > 0$) times something > 1 without factors of 2 or 5 have *eventually* periodic (but not purely periodic) decimals:

$$\frac{3}{44} = \frac{3}{2^2 \cdot 11} = \frac{5^2 \cdot 3}{5^2 \cdot 2^2 \cdot 11} = \frac{1}{100}\frac{75}{11} = \frac{1}{100}\left(6 + \frac{9}{11}\right) = .06 + ?.$$

(split $\frac{75}{11}$ into integer and fractional parts) **Repeat the method.**

$$11 \cdot 9 = 99 \Rightarrow \frac{9}{11} = \frac{9 \cdot 9}{99} = \frac{81}{99} = .\overline{81},$$

so

Fractions with a denominator that is $2^i 5^j$ ($i > 0$ or $j > 0$) times something > 1 without factors of 2 or 5 have *eventually* periodic (but not purely periodic) decimals:

$$\frac{3}{44} = \frac{3}{2^2 \cdot 11} = \frac{5^2 \cdot 3}{5^2 \cdot 2^2 \cdot 11} = \frac{1}{100}\frac{75}{11} = \frac{1}{100}\left(6 + \frac{9}{11}\right) = .06 + ?.$$

(split $\frac{75}{11}$ into integer and fractional parts) **Repeat the method.**

$$11 \cdot 9 = 99 \Rightarrow \frac{9}{11} = \frac{9 \cdot 9}{99} = \frac{81}{99} = .\overline{81},$$

so (again no carrying!) we recover the other part of Exercise 1(ii):

$$\frac{3}{44} = .06 + \frac{1}{100}\frac{9}{11} = .06 + \frac{1}{100}.\overline{81} = .06 + .00\overline{81} = .06\overline{81}.$$

Fractions are Periodic Decimals: Eventual Periodicity

Fractions with a denominator that is $2^i 5^j$ ($i > 0$ or $j > 0$) times something > 1 without factors of 2 or 5 have *eventually* periodic (but not purely periodic) decimals:

$$\frac{3}{44} = \frac{3}{2^2 \cdot 11} = \frac{5^2 \cdot 3}{5^2 \cdot 2^2 \cdot 11} = \frac{1}{100}\frac{75}{11} = \frac{1}{100}\left(6 + \frac{9}{11}\right) = .06 + ?.$$

(split $\frac{75}{11}$ into integer and fractional parts) **Repeat the method.**

$$11 \cdot 9 = 99 \Rightarrow \frac{9}{11} = \frac{9 \cdot 9}{99} = \frac{81}{99} = .\overline{81},$$

so (again no carrying!) we recover the other part of Exercise 1(ii):

$$\frac{3}{44} = .06 + \frac{1}{100}\frac{9}{11} = .06 + \frac{1}{100}.\overline{81} = .06 + .00\overline{81} = .06\overline{81}.$$

Method: absorb 2's and 5's in denominator into a power of 10; this creates eventually (but not purely) periodic decimals.

Fractions are Periodic Decimals: Eventual Periodicity

We can run “eventually periodic” examples in reverse, going from the decimal to a fraction (that we can try to put in reduced form).

We can run "eventually periodic" examples in reverse, going from the decimal to a fraction (that we can try to put in reduced form). **Let's try with the preceding example**:

$$\begin{aligned} x &= .06\overline{81} \\ &= .06818181\ldots \end{aligned}$$

We can run "eventually periodic" examples in reverse, going from the decimal to a fraction (that we can try to put in reduced form). **Let's try with the preceding example**:

$$\begin{aligned} x &= .06\overline{81} \\ &= .06818181\ldots \\ 10x &= .6818181\ldots \end{aligned}$$

We can run “eventually periodic” examples in reverse, going from the decimal to a fraction (that we can try to put in reduced form). **Let’s try with the preceding example**:

$$\begin{aligned} x &= .06\overline{81} \\ &= .06818181\ldots \\ 10x &= .6818181\ldots \\ 10^2x &= 6.\color{red}{818181\ldots} \end{aligned}$$

Fractions are Periodic Decimals: Eventual Periodicity

We can run "eventually periodic" examples in reverse, going from the decimal to a fraction (that we can try to put in reduced form). **Let's try with the preceding example**:

$$\begin{aligned} x &= .06\overline{81} \\ &= .06818181\ldots \\ 10x &= .6818181\ldots \\ 10^2x &= 6.818181\ldots \\ 10^4x &= 681.818181\ldots \end{aligned}$$

We can run "eventually periodic" examples in reverse, going from the decimal to a fraction (that we can try to put in reduced form). **Let's try with the preceding example**:

$$
\begin{aligned}
x &= .06\overline{81} \\
&= .06818181\ldots \\
10x &= .6818181\ldots \\
10^2x &= 6.818181\ldots \\
10^4x &= 681.818181\ldots \\
\text{Subtract:}\quad (10^4 - 10^2)x &= 681 - 6 \\
9900x &= 675
\end{aligned}
$$

Fractions are Periodic Decimals: Eventual Periodicity

We can run "eventually periodic" examples in reverse, going from the decimal to a fraction (that we can try to put in reduced form). **Let's try with the preceding example**:

$$\begin{aligned}
x &= .06\overline{81} \\
&= .06818181\ldots \\
10x &= .6818181\ldots \\
10^2x &= 6.818181\ldots \\
10^4x &= 681.818181\ldots \\
\text{Subtract:}\quad (10^4 - 10^2)x &= 681 - 6 \\
9900x &= 675 \\
x &= \frac{675}{9900}
\end{aligned}$$

Fractions are Periodic Decimals: Eventual Periodicity

We can run "eventually periodic" examples in reverse, going from the decimal to a fraction (that we can try to put in reduced form). **Let's try with the preceding example**:

$$\begin{aligned} x &= .06\overline{81} \\ &= .06818181\ldots \\ 10x &= .6818181\ldots \\ 10^2x &= 6.818181\ldots \\ 10^4x &= 681.818181\ldots \\ \text{Subtract:}\quad (10^4-10^2)x &= 681-6 \\ 9900x &= 675 \\ x &= \frac{675}{9900} \end{aligned}$$

Since $9900 = 99 \cdot 100 = (3^2 \cdot 11) \cdot (2^2 \cdot 5^2)$ and one finds that $3^2 \cdot 5^2$ is a factor of 675, this yields $x = 3/44$ in reduced form.

Summary So Far

We have shown that eventually periodic decimals (allowing for a 0-string) are **exactly** the decimal expansions of fractions, with the possibilities falling into three cases:

Decimal Type	Denominator of the Fraction
Finite	Only 2's and 5's
Purely Periodic (not finite)	No 2's or 5's
Eventually (not purely) Periodic	2's or 5's (or both) times more

$$
\begin{aligned}
.38 &= \frac{19}{2 \cdot 5^2} \\
.\overline{518} &= \frac{14}{27} \\
.4\overline{380952} &= \frac{46}{5 \cdot 21}
\end{aligned}
$$

Summary So Far

We have shown that eventually periodic decimals (allowing for a 0-string) are **exactly** the decimal expansions of fractions, with the possibilities falling into three cases:

Decimal Type	Denominator of the Fraction
Finite	Only 2's and 5's
Purely Periodic (not finite)	No 2's or 5's
Eventually (not purely) Periodic	2's or 5's (or both) times more

$$
\begin{aligned}
.38 &= \frac{19}{2 \cdot 5^2} \\
.\overline{518} &= \frac{14}{27} \\
.4\overline{380952} &= \frac{46}{5 \cdot 21}
\end{aligned}
$$

Let's work out another example of an eventually (but not purely) periodic decimal for a fraction...

Example: Decimal Expansion of $53/82$

The reduced fraction $53/82$ has even denominator, so won't be purely periodic.

Example: Decimal Expansion of 53/82

The reduced fraction 53/82 has even denominator, so won't be purely periodic. **Compute it via our general method**:

$$\frac{53}{82} = \frac{53}{2 \cdot 41} =$$

Example: Decimal Expansion of $53/82$

The reduced fraction $53/82$ has even denominator, so won't be purely periodic. **Compute it via our general method**:

$$\frac{53}{82} = \frac{53}{2 \cdot 41} = \frac{5 \cdot 53}{5 \cdot 2 \cdot 41} = \frac{1}{10}\frac{265}{41} = \frac{1}{10}\left(6 + \frac{19}{41}\right).$$

(We split $265/41$ into integer and fractional parts so no carrying below.)

Example: Decimal Expansion of 53/82

The reduced fraction 53/82 has even denominator, so won't be purely periodic. **Compute it via our general method**:

$$\frac{53}{82} = \frac{53}{2 \cdot 41} = \frac{5 \cdot 53}{5 \cdot 2 \cdot 41} = \frac{1}{10}\frac{265}{41} = \frac{1}{10}\left(6 + \frac{19}{41}\right).$$

(We split 265/41 into integer and fractional parts so no carrying below.) This has an eventually periodic decimal $.6\overline{c_1 c_2 \ldots c_d}$.

Example: Decimal Expansion of 53/82

The reduced fraction 53/82 has even denominator, so won't be purely periodic. **Compute it via our general method**:

$$\frac{53}{82} = \frac{53}{2 \cdot 41} = \frac{5 \cdot 53}{5 \cdot 2 \cdot 41} = \frac{1}{10}\frac{265}{41} = \frac{1}{10}\left(6 + \frac{19}{41}\right).$$

(We split 265/41 into integer and fractional parts so no carrying below.) This has an eventually periodic decimal $.6\overline{c_1 c_2 \ldots c_d}$.

To get decimal for 19/41, compute $10^d - 1$ for $d = 1, 2, \ldots, 40$ until it's divisible by 41: $10^5 - 1 = 41 \cdot 2439$ (cf. Exer. 2(i)), so

Example: Decimal Expansion of $53/82$

The reduced fraction $53/82$ has even denominator, so won't be purely periodic. **Compute it via our general method**:

$$\frac{53}{82} = \frac{53}{2 \cdot 41} = \frac{5 \cdot 53}{5 \cdot 2 \cdot 41} = \frac{1}{10}\frac{265}{41} = \frac{1}{10}\left(6 + \frac{19}{41}\right).$$

(We split $265/41$ into integer and fractional parts so no carrying below.) This has an eventually periodic decimal $.6\overline{c_1 c_2 \dots c_d}$.

To get decimal for $19/41$, compute $10^d - 1$ for $d = 1, 2, \dots, 40$ until it's divisible by 41: $10^5 - 1 = 41 \cdot 2439$ (cf. Exer. 2(i)), so

$$\frac{19}{41} = \frac{19 \cdot 2439}{41 \cdot 2439} = \frac{46341}{99999} = .\overline{46341}$$

(as done for Exercise 2(ii)).

Example: Decimal Expansion of 53/82

The reduced fraction 53/82 has even denominator, so won't be purely periodic. **Compute it via our general method**:

$$\frac{53}{82} = \frac{53}{2 \cdot 41} = \frac{5 \cdot 53}{5 \cdot 2 \cdot 41} = \frac{1}{10}\frac{265}{41} = \frac{1}{10}\left(6 + \frac{19}{41}\right).$$

(We split 265/41 into integer and fractional parts so no carrying below.) This has an eventually periodic decimal $.6\overline{c_1 c_2 \ldots c_d}$.

To get decimal for 19/41, compute $10^d - 1$ for $d = 1, 2, \ldots, 40$ until it's divisible by 41: $10^5 - 1 = 41 \cdot 2439$ (cf. Exer. 2(i)), so

$$\frac{19}{41} = \frac{19 \cdot 2439}{41 \cdot 2439} = \frac{46341}{99999} = .\overline{46341}$$

(as done for Exercise 2(ii)). Therefore

$$\frac{53}{82} = \frac{1}{10}\left(6 + \frac{19}{41}\right) = \frac{1}{10}\left(6 + .\overline{46341}\right) = .6\overline{46341}.$$

(As usual, no carrying occurred!)

We now understand the reasons for existence of (eventually) periodic structure in the decimal expansions of fractions. There are more refined *structural* questions about this, such as:

Patterns in Purely Periodic Decimals

We now understand the reasons for existence of (eventually) periodic structure in the decimal expansions of fractions. There are more refined *structural* questions about this, such as:

Question: How are the period length and periodic part of a *purely periodic* decimal related to the "fractional form" of the number (e.g., why do periodic parts of $1/7, 2/7, \ldots, 6/7$ cycle around)?

Patterns in Purely Periodic Decimals

We now understand the reasons for existence of (eventually) periodic structure in the decimal expansions of fractions. There are more refined *structural* questions about this, such as:

Question: How are the period length and periodic part of a *purely periodic* decimal related to the "fractional form" of the number (e.g., why do periodic parts of $1/7, 2/7, \ldots, 6/7$ cycle around)?

Preliminary work by Lambert, Bernoulli, Hindenburg (ca. 1750–1770). The whole topic was systematically explained by Gauss (Disquisitiones Arithmeticae, 1801).

Patterns in Purely Periodic Decimals

We now understand the reasons for existence of (eventually) periodic structure in the decimal expansions of fractions. There are more refined *structural* questions about this, such as:

Question: How are the period length and periodic part of a *purely periodic* decimal related to the "fractional form" of the number (e.g., why do periodic parts of $1/7, 2/7, \ldots, 6/7$ cycle around)?

Preliminary work by Lambert, Bernoulli, Hindenburg (ca. 1750–1770). The whole topic was systematically explained by Gauss (Disquisitiones Arithmeticae, 1801).

When n is not divisible by 2 or 5, set

$$L(n) = \text{ period length of the decimal for } \frac{1}{n}.$$

Patterns in Purely Periodic Decimals

We now understand the reasons for existence of (eventually) periodic structure in the decimal expansions of fractions. There are more refined *structural* questions about this, such as:

Question: How are the period length and periodic part of a *purely periodic* decimal related to the "fractional form" of the number (e.g., why do periodic parts of $1/7, 2/7, \ldots, 6/7$ cycle around)?

Preliminary work by Lambert, Bernoulli, Hindenburg (ca. 1750–1770). The whole topic was systematically explained by Gauss (Disquisitiones Arithmeticae, 1801).

When n is not divisible by 2 or 5, set

$$L(n) = \text{ period length of the decimal for } \frac{1}{n}.$$

Example

$$\frac{1}{3} = .\overline{3} \Rightarrow L(3) = 1, \quad \frac{1}{7} = .\overline{142857} \Rightarrow L(7) = 6.$$

Variation of Periods Looks Irregular

n	Decimal for $1/n$	$L(n)$
21	$.\overline{047619}$	6
23	$.\overline{0434782608695652173913}$	22
27	$.\overline{037}$	3
29	$.\overline{0344827586206896551724137931}$	28
31	$.\overline{032258064516129}$	15
33	$.\overline{03}$	2
37	$.\overline{027}$ (Exer. 2)	3
39	$.\overline{025641}$	6
41	$.\overline{02439}$	5
43	$.\overline{023255813953488372093}$	21

Variation of Periods Looks Irregular

n	Decimal for $1/n$	$L(n)$
21	$.\overline{047619}$	6
23	$.\overline{0434782608695652173913}$	22
27	$.\overline{037}$	3
29	$.\overline{0344827586206896551724137931}$	28
31	$.\overline{032258064516129}$	15
33	$.\overline{03}$	2
37	$.\overline{027}$ (Exer. 2)	3
39	$.\overline{025641}$	6
41	$.\overline{02439}$	5
43	$.\overline{023255813953488372093}$	21

Key insight of Gauss: The structure of a fraction's decimal expansion depends on the *prime factorization* of the denominator (not on the fraction's position in the number line), and on specific properties of those primes. This is surprising!

Period of $1/p$, p prime, $p \neq 2$ or 5

p	3	7	11	13	17	19	23	29	31	37
$L(p)$	1	6	2	6	16	18	22	28	15	3
p	41	43	47	53	59	61	67	71	73	79
$L(p)$	5	21	46	13	58	60	33	35	8	13
p	83	89	97	101	103	107	109	113	127	131
$L(p)$	41	44	96	4	34	53	108	112	42	130

For $p = 7, 13, 37, 41$, $L(p)$ is what we found in Exer. 2(i): the least d with $10^d - 1$ a multiple of p. **Why?**

Period of $1/p$, p prime, $p \neq 2$ or 5

p	3	7	11	13	17	19	23	29	31	37
$L(p)$	1	6	2	6	16	18	22	28	15	3
p	41	43	47	53	59	61	67	71	73	79
$L(p)$	5	21	46	13	58	60	33	35	8	13
p	83	89	97	101	103	107	109	113	127	131
$L(p)$	41	44	96	4	34	53	108	112	42	130

For $p = 7, 13, 37, 41$, $L(p)$ is what we found in Exer. 2(i): the least d with $10^d - 1$ a multiple of p. **Why?**

Does the data suggest any general properties for $L(p)$?

Period of $1/p$, p prime, $p \neq 2$ or 5

p	3	7	11	13	17	19	23	29	31	37
$L(p)$	1	6	2	6	16	18	22	28	15	3
p	41	43	47	53	59	61	67	71	73	79
$L(p)$	5	21	46	13	58	60	33	35	8	13
p	83	89	97	101	103	107	109	113	127	131
$L(p)$	41	44	96	4	34	53	108	112	42	130

For $p = 7, 13, 37, 41$, $L(p)$ is what we found in Exer. 2(i): the least d with $10^d - 1$ a multiple of p. **Why?**

Does the data suggest any general properties for $L(p)$?

- Often $L(p) = p - 1$: proportion in table is $12/30 \approx .4$.
  (We will come back to this frequency issue at the very end.)

Period of $1/p$, p prime, $p \neq 2$ or 5

p	3	7	11	13	17	19	23	29	31	37
$L(p)$	1	6	2	6	16	18	22	28	15	3
p	41	43	47	53	59	61	67	71	73	79
$L(p)$	5	21	46	13	58	60	33	35	8	13
p	83	89	97	101	103	107	109	113	127	131
$L(p)$	41	44	96	4	34	53	108	112	42	130

For $p = 7, 13, 37, 41$, $L(p)$ is what we found in Exer. 2(i): the least d with $10^d - 1$ a multiple of p. **Why?**

Does the data suggest any general properties for $L(p)$?

- Often $L(p) = p - 1$: proportion in table is $12/30 \approx .4$.
  (We will come back to this frequency issue at the very end.)
- $L(p)$ is a factor of $p - 1$ without exceptions.

Period of $1/p$, p prime, $p \neq 2$ or 5

p	3	7	11	13	17	19	23	29	31	37
$L(p)$	1	6	2	6	16	18	22	28	15	3
p	41	43	47	53	59	61	67	71	73	79
$L(p)$	5	21	46	13	58	60	33	35	8	13
p	83	89	97	101	103	107	109	113	127	131
$L(p)$	41	44	96	4	34	53	108	112	42	130

For $p = 7, 13, 37, 41$, $L(p)$ is what we found in Exer. 2(i): the least d with $10^d - 1$ a multiple of p. **Why?**

Does the data suggest any general properties for $L(p)$?

- Often $L(p) = p - 1$: proportion in table is $12/30 \approx .4$. (We will come back to this frequency issue at the very end.)
- $L(p)$ is a factor of $p - 1$ without exceptions.

To understand this, let's look at decimal expansions of all *reduced* fractions a/n with $0 < a < n$ and n not divisible by 2 or 5.

Comparing Decimal Digits for Common Denominators

The decimal expansions of reduced fractions $a/7$ with $0 < a < 7$ (called *proper*) have the same *cycle* of digits: 142857, up to the choice of which digit comes first.

Fraction	Decimal
$1/7$	$.\overline{142857}$
$2/7$	$.\overline{285714}$
$3/7$	$.\overline{428571}$
$4/7$	$.\overline{571428}$
$5/7$	$.\overline{714285}$
$6/7$	$.\overline{857142}$

The decimal expansions of reduced fractions $a/7$ with $0 < a < 7$ (called *proper*) have the same *cycle* of digits: 142857, up to the choice of which digit comes first.

Fraction	Decimal
$1/7$	$.\overline{142857}$
$2/7$	$.\overline{285714}$
$3/7$	$.\overline{428571}$
$4/7$	$.\overline{571428}$
$5/7$	$.\overline{714285}$
$6/7$	$.\overline{857142}$

Let's look at some more examples with small denominator...

Comparing Decimal Digits for Common Denominators

Consider reduced proper fractions a/n with $n = 3, 7, 9$:

Fraction	Decimal	Fraction	Decimal	Number of Cycles
$1/3$	$.\overline{3}$	$2/3$	$.\overline{6}$	2
$1/7$ $3/7$ $5/7$	$.\overline{142857}$ $.\overline{428571}$ $.\overline{714285}$	$2/7$ $4/7$ $6/7$	$.\overline{285714}$ $.\overline{571428}$ $.\overline{857142}$	1
$1/9$ $4/9$ $7/9$	$.\overline{1}$ $.\overline{4}$ $.\overline{7}$	$2/9$ $5/9$ $8/9$	$.\overline{2}$ $.\overline{5}$ $.\overline{8}$	6

For $n = 3, 9$ there's no "cyclic pattern" but at least the *length* of the period is the same for all a/n as we vary a.

Comparing Decimal Digits for Common Denominators

Consider reduced proper fractions a/n with $n = 3, 7, 9$:

Fraction	Decimal	Fraction	Decimal	Number of Cycles
$1/3$	$.\overline{3}$	$2/3$	$.\overline{6}$	2
$1/7$ $3/7$ $5/7$	$.\overline{142857}$ $.\overline{428571}$ $.\overline{714285}$	$2/7$ $4/7$ $6/7$	$.\overline{285714}$ $.\overline{571428}$ $.\overline{857142}$	1
$1/9$ $4/9$ $7/9$	$.\overline{1}$ $.\overline{4}$ $.\overline{7}$	$2/9$ $5/9$ $8/9$	$.\overline{2}$ $.\overline{5}$ $.\overline{8}$	6

For $n = 3, 9$ there's no "cyclic pattern" but at least the *length* of the period is the same for all a/n as we vary a.

Let's work on Exercise 3, exploring denominators 11, 13, 17.

Denominators 11 and 13 (solution to Exercise 3(i), (ii)):

Fraction	Decimal	Fraction	Decimal	Number of Cycles
1/11	$.\overline{09}$	2/11	$.\overline{18}$	
3/11	$.\overline{27}$	4/11	$.\overline{36}$	
5/11	$.\overline{45}$	6/11	$.\overline{54}$	
7/11	$.\overline{63}$	8/11	$.\overline{72}$	
9/11	$.\overline{81}$	10/11	$.\overline{90}$	
1/13	$.\overline{076923}$	2/13	$.\overline{153846}$	
3/13	$.\overline{230769}$	4/13	$.\overline{307692}$	
5/13	$.\overline{384615}$	6/13	$.\overline{461538}$	
7/13	$.\overline{538461}$	8/13	$.\overline{615384}$	
9/13	$.\overline{692307}$	10/13	$.\overline{769230}$	
11/13	$.\overline{846153}$	12/13	$.\overline{923076}$	

Denominators 11 and 13 (solution to Exercise 3(i), (ii)):

Fraction	Decimal	Fraction	Decimal	Number of Cycles
1/11	$.\overline{09}$	2/11	$.\overline{18}$	5
3/11	$.\overline{27}$	4/11	$.\overline{36}$	
5/11	$.\overline{45}$	6/11	$.\overline{54}$	
7/11	$.\overline{63}$	8/11	$.\overline{72}$	
9/11	$.\overline{81}$	10/11	$.\overline{90}$	
1/13	$.\overline{076923}$	2/13	$.\overline{153846}$	
3/13	$.\overline{230769}$	4/13	$.\overline{307692}$	
5/13	$.\overline{384615}$	6/13	$.\overline{461538}$	
7/13	$.\overline{538461}$	8/13	$.\overline{615384}$	
9/13	$.\overline{692307}$	10/13	$.\overline{769230}$	
11/13	$.\overline{846153}$	12/13	$.\overline{923076}$	

Comparing Decimal Digits for Common Denominators

Denominators 11 and 13 (solution to Exercise 3(i), (ii)):

Fraction	Decimal	Fraction	Decimal	Number of Cycles
1/11	$.\overline{09}$	2/11	$.\overline{18}$	5
3/11	$.\overline{27}$	4/11	$.\overline{36}$	
5/11	$.\overline{45}$	6/11	$.\overline{54}$	
7/11	$.\overline{63}$	8/11	$.\overline{72}$	
9/11	$.\overline{81}$	10/11	$.\overline{90}$	
1/13	$.\overline{076923}$	2/13	$.\overline{153846}$	2
3/13	$.\overline{230769}$	4/13	$.\overline{307692}$	
5/13	$.\overline{384615}$	6/13	$.\overline{461538}$	
7/13	$.\overline{538461}$	8/13	$.\overline{615384}$	
9/13	$.\overline{692307}$	10/13	$.\overline{769230}$	
11/13	$.\overline{846153}$	12/13	$.\overline{923076}$	

Denominators 11 and 13 (solution to Exercise 3(i), (ii)):

Fraction	Decimal	Fraction	Decimal	Number of Cycles
1/11	$.\overline{09}$	2/11	$.\overline{18}$	5
3/11	$.\overline{27}$	4/11	$.\overline{36}$	(2 digits each)
5/11	$.\overline{45}$	6/11	$.\overline{54}$	(10 fractions)
7/11	$.\overline{63}$	8/11	$.\overline{72}$	
9/11	$.\overline{81}$	10/11	$.\overline{90}$	
1/13	$.\overline{076923}$	2/13	$.\overline{153846}$	2
3/13	$.\overline{230769}$	4/13	$.\overline{307692}$	(6 digits each)
5/13	$.\overline{384615}$	6/13	$.\overline{461538}$	(12 fractions)
7/13	$.\overline{538461}$	8/13	$.\overline{615384}$	
9/13	$.\overline{692307}$	10/13	$.\overline{769230}$	
11/13	$.\overline{846153}$	12/13	$.\overline{923076}$	

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	
3/17		4/17	
5/17		6/17	
7/17		8/17	
9/17		10/17	
11/17		12/17	
13/17		14/17	
15/17		16/17	

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	
3/17		4/17	
5/17		6/17	
7/17		8/17	
9/17		10/17	
11/17		12/17	$.\overline{7058823529411764}$
13/17		14/17	
15/17		16/17	

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	
3/17		4/17	
5/17		6/17	
7/17		8/17	$.\overline{4705882352941176}$
9/17		10/17	
11/17		12/17	$.\overline{7058823529411764}$
13/17		14/17	
15/17		16/17	

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	
3/17		4/17	
5/17		6/17	
7/17		8/17	$.\overline{4705882352941176}$
9/17		10/17	
11/17	$.\overline{6470588235294117}$	12/17	$.\overline{7058823529411764}$
13/17		14/17	
15/17		16/17	

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	
3/17		4/17	
5/17		6/17	
7/17		8/17	$.\overline{4705882352941176}$
9/17		10/17	
11/17	$.\overline{6470588235294117}$	12/17	$.\overline{7058823529411764}$
13/17	$.\overline{7647058823529411}$	14/17	
15/17		16/17	

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	
3/17	$.\overline{1764705882352941}$	4/17	
5/17		6/17	
7/17		8/17	$.\overline{4705882352941176}$
9/17		10/17	
11/17	$.\overline{6470588235294117}$	12/17	$.\overline{7058823529411764}$
13/17	$.\overline{7647058823529411}$	14/17	
15/17		16/17	

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	$.\overline{1176470588235294}$
3/17	$.\overline{1764705882352941}$	4/17	
5/17		6/17	
7/17		8/17	$.\overline{4705882352941176}$
9/17		10/17	
11/17	$.\overline{6470588235294117}$	12/17	$.\overline{7058823529411764}$
13/17	$.\overline{7647058823529411}$	14/17	
15/17		16/17	

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	$.\overline{1176470588235294}$
3/17	$.\overline{1764705882352941}$	4/17	
5/17		6/17	
7/17	$.\overline{4117647058823529}$	8/17	$.\overline{4705882352941176}$
9/17		10/17	
11/17	$.\overline{6470588235294117}$	12/17	$.\overline{7058823529411764}$
13/17	$.\overline{7647058823529411}$	14/17	
15/17		16/17	

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	$.\overline{1176470588235294}$
3/17	$.\overline{1764705882352941}$	4/17	
5/17		6/17	
7/17	$.\overline{4117647058823529}$	8/17	$.\overline{4705882352941176}$
9/17		10/17	
11/17	$.\overline{6470588235294117}$	12/17	$.\overline{7058823529411764}$
13/17	$.\overline{7647058823529411}$	14/17	
15/17		16/17	$.\overline{9411764705882352}$

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	$.\overline{1176470588235294}$
3/17	$.\overline{1764705882352941}$	4/17	
5/17	$.\overline{2941176470588235}$	6/17	
7/17	$.\overline{4117647058823529}$	8/17	$.\overline{4705882352941176}$
9/17		10/17	
11/17	$.\overline{6470588235294117}$	12/17	$.\overline{7058823529411764}$
13/17	$.\overline{7647058823529411}$	14/17	
15/17		16/17	$.\overline{9411764705882352}$

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	$.\overline{1176470588235294}$
3/17	$.\overline{1764705882352941}$	4/17	
5/17	$.\overline{2941176470588235}$	6/17	
7/17	$.\overline{4117647058823529}$	8/17	$.\overline{4705882352941176}$
9/17	$.\overline{5294117647058823}$	10/17	
11/17	$.\overline{6470588235294117}$	12/17	$.\overline{7058823529411764}$
13/17	$.\overline{7647058823529411}$	14/17	
15/17		16/17	$.\overline{9411764705882352}$

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	$.\overline{1176470588235294}$
3/17	$.\overline{1764705882352941}$	4/17	
5/17	$.\overline{2941176470588235}$	6/17	$.\overline{3529411764705882}$
7/17	$.\overline{4117647058823529}$	8/17	$.\overline{4705882352941176}$
9/17	$.\overline{5294117647058823}$	10/17	
11/17	$.\overline{6470588235294117}$	12/17	$.\overline{7058823529411764}$
13/17	$.\overline{7647058823529411}$	14/17	
15/17		16/17	$.\overline{9411764705882352}$

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
$1/17$	$.\overline{0588235294117647}$	$2/17$	$.\overline{1176470588235294}$
$3/17$	$.\overline{1764705882352941}$	$4/17$	$.\overline{2352941176470588}$
$5/17$	$.\overline{2941176470588235}$	$6/17$	$.\overline{3529411764705882}$
$7/17$	$.\overline{4117647058823529}$	$8/17$	$.\overline{4705882352941176}$
$9/17$	$.\overline{5294117647058823}$	$10/17$	
$11/17$	$.\overline{6470588235294117}$	$12/17$	$.\overline{7058823529411764}$
$13/17$	$.\overline{7647058823529411}$	$14/17$	
$15/17$		$16/17$	$.\overline{9411764705882352}$

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	$.\overline{1176470588235294}$
3/17	$.\overline{1764705882352941}$	4/17	$.\overline{2352941176470588}$
5/17	$.\overline{2941176470588235}$	6/17	$.\overline{3529411764705882}$
7/17	$.\overline{4117647058823529}$	8/17	$.\overline{4705882352941176}$
9/17	$.\overline{5294117647058823}$	10/17	
11/17	$.\overline{6470588235294117}$	12/17	$.\overline{7058823529411764}$
13/17	$.\overline{7647058823529411}$	14/17	$.\overline{8235294117647058}$
15/17		16/17	$.\overline{9411764705882352}$

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	$.\overline{1176470588235294}$
3/17	$.\overline{1764705882352941}$	4/17	$.\overline{2352941176470588}$
5/17	$.\overline{2941176470588235}$	6/17	$.\overline{3529411764705882}$
7/17	$.\overline{4117647058823529}$	8/17	$.\overline{4705882352941176}$
9/17	$.\overline{5294117647058823}$	10/17	
11/17	$.\overline{6470588235294117}$	12/17	$.\overline{7058823529411764}$
13/17	$.\overline{7647058823529411}$	14/17	$.\overline{8235294117647058}$
15/17	$.\overline{8823529411764705}$	16/17	$.\overline{9411764705882352}$

Comparing Decimal Digits for Common Denominators

For denominator 17, let's fill in where cyclic shifts of expansion of 1/17 occur, one shift at a time (also solving Exercise 3(iii)):

Fraction	Decimal	Fraction	Decimal
1/17	$.\overline{0588235294117647}$	2/17	$.\overline{1176470588235294}$
3/17	$.\overline{1764705882352941}$	4/17	$.\overline{2352941176470588}$
5/17	$.\overline{2941176470588235}$	6/17	$.\overline{3529411764705882}$
7/17	$.\overline{4117647058823529}$	8/17	$.\overline{4705882352941176}$
9/17	$.\overline{5294117647058823}$	10/17	$.\overline{5882352941176470}$
11/17	$.\overline{6470588235294117}$	12/17	$.\overline{7058823529411764}$
13/17	$.\overline{7647058823529411}$	14/17	$.\overline{8235294117647058}$
15/17	$.\overline{8823529411764705}$	16/17	$.\overline{9411764705882352}$

All 16 of these decimals form 1 cycle (with 16 digits). Let's try another example...

Comparing Decimal Digits for Common Denominators

Denominator 21: **Pick a color and compute the periodic parts.**

Fraction	Decimal	Fraction	Decimal
1/21		2/21	
4/21		5/21	
8/21		10/21	
11/21		13/21	
16/21		17/21	
19/21		20/21	

Comparing Decimal Digits for Common Denominators

Denominator 21:

Fraction	Decimal	Fraction	Decimal
1/21	$.\overline{047619}$	2/21	$.\overline{095238}$
4/21	$.\overline{190476}$	5/21	$.\overline{238095}$
8/21	$.\overline{380952}$	10/21	$.\overline{476190}$
11/21	$.\overline{523809}$	13/21	$.\overline{619047}$
16/21	$.\overline{761904}$	17/21	$.\overline{809523}$
19/21	$.\overline{904761}$	20/21	$.\overline{952380}$

There are 2 cycles, 6 digits each, and 12 proper fractions overall. This suggests trying to extend the pattern of Exercise 3(ii) to non-prime denominators (not divisible by 2 or 5).

Let's try another...

Comparing Decimal Digits for Common Denominators

Denominator 27.

Fraction	Decimal	Fraction	Decimal	Fraction	Decimal
1/27	$.\overline{037}$	2/27	$.\overline{074}$	4/27	$.\overline{148}$
5/27	$.\overline{185}$	7/27	$.\overline{259}$	8/27	$.\overline{296}$
10/27	$.\overline{370}$	11/27	$.\overline{407}$	13/27	$.\overline{481}$
14/27	$.\overline{518}$	16/27	$.\overline{592}$	17/27	$.\overline{629}$
19/27	$.\overline{703}$	20/27	$.\overline{740}$	22/27	$.\overline{814}$
23/27	$.\overline{851}$	25/27	$.\overline{925}$	26/27	$.\overline{962}$

There are 6 cycles, 3 digits each, and 18 proper fractions overall.

Can we formulate a general conjecture concerning decimals for reduced proper a/n with n not divisible by 2 or 5 (i.e., a/n has a purely periodic decimal)?

Can we formulate a general conjecture concerning decimals for reduced proper a/n with n not divisible by 2 or 5 (i.e., a/n has a purely periodic decimal)?

Conjecture

For n not divisible by 2 or 5, all reduced proper fractions with denominator n have the **same** *decimal period length $L(n)$. Moreover,*

$$L(n) \cdot \#(\textit{cycles}) = \#\textit{reduced proper fractions with denominator } n.$$

Unifying Previous Data

Can we formulate a general conjecture concerning decimals for reduced proper a/n with n not divisible by 2 or 5 (i.e., a/n has a purely periodic decimal)?

Conjecture

For n not divisible by 2 or 5, all reduced proper fractions with denominator n have the **same** *decimal period length $L(n)$. Moreover,*

$$L(n) \cdot \#(\textit{cycles}) = \#\textit{reduced proper fractions with denominator } n.$$

Does this illuminate the case $n = p$ a prime ($\neq 2, 5$)?

Can we formulate a general conjecture concerning decimals for reduced proper a/n with n not divisible by 2 or 5 (i.e., a/n has a purely periodic decimal)?

Conjecture

For n not divisible by 2 or 5, all reduced proper fractions with denominator n have the **same** *decimal period length $L(n)$. Moreover,*

$$L(n) \cdot \#(\mathit{cycles}) = \#\mathit{reduced\ proper\ fractions\ with\ denominator}\ n.$$

Does this illuminate the case $n = p$ a prime ($\neq 2, 5$)?

Example

For a prime $p \neq 2, 5$, there are $p - 1$ reduced proper fractions with denominator p (i.e., $1/p, 2/p, \ldots, (p-1)/p$), so the Conjecture, if true, implies $L(p)$ is a factor of $p - 1$. That would explain an earlier observation in the tables.

Theorem (Gauss, 1801)

For n not divisible by 2 or 5,

1. *all reduced proper fractions with denominator n have the* **same** *decimal period length $L(n)$,*
2. $L(n) \cdot \#(\textit{cycles}) = \#$*reduced proper frac. with denominator n.*

Proof of (1):

Theorem (Gauss, 1801)

For n not divisible by 2 or 5,

1. *all reduced proper fractions with denominator n have the* **same** *decimal period length $L(n)$,*
2. $L(n) \cdot \#(\textit{cycles}) = \#\textit{reduced proper frac. with denominator } n.$

Proof of (1): We saw that a *reduced* a/n has a d-digit decimal periodic part precisely when d is minimal such that n is a factor of $10^d - 1$, a condition that has **nothing to do** with a.

Theorem (Gauss, 1801)

For n not divisible by 2 or 5,

1. *all reduced proper fractions with denominator n have the* **same** *decimal period length $L(n)$,*
2. $L(n) \cdot \#(cycles) = \#reduced\ proper\ frac.\ with\ denominator\ n.$

Proof of (1): We saw that a *reduced* a/n has a d-digit decimal periodic part precisely when d is minimal such that n is a factor of $10^d - 1$, a condition that has **nothing to do** with a.
The proof of (2) involves a more profound idea, as we'll see.

Here is part (2) on its own:

Theorem (Gauss)

For n not divisible by 2 or 5,

$$L(n) \cdot \#(\textit{cycles}) = \#\textit{reduced proper fractions with denominator } n.$$

n	$L(n)$	#Cycles	# Red. Fractions
3	1	2	2
7	6	1	6
9	1	6	6
11	2	5	10
13	6	2	12
17	16	1	16
21	6	2	12
27	3	6	18

Unifying Previous Data

Here is part (2) on its own:

Theorem (Gauss)

For n not divisible by 2 or 5,

$$L(n) \cdot \#(\text{cycles}) = \#\text{reduced proper fractions with denominator } n.$$

n	$L(n)$	#Cycles	# Red. Fractions
3	1	2	2
7	6	1	6
9	1	6	6
11	2	5	10
13	6	2	12
17	16	1	16
21	6	2	12
27	3	6	18

The proof requires "periodic math" (usually called "modular arithmetic"). It is Gauss' most important insight on this topic.

Periodic Math (Clock Arithmetic)

We will consider as equal all numbers that are in the same relative position between multiples of a fixed positive integer (such as 12).

We will consider as equal all numbers that are in the same relative position between multiples of a fixed positive integer (such as 12).

Example (Time on a clock)

$10 + 4 = 2$ and $4 - 7 = 9$. The numbers 14 and 2 are both 2 units to the right of a multiple of 12 and -3 and 9 are both 3 units to the left of a multiple of 12. Here we proceed as if $12 = 0$.

Periodic Math (Clock Arithmetic)

We will consider as equal all numbers that are in the same relative position between multiples of a fixed positive integer (such as 12).

Example (Time on a clock)

$10 + 4 = 2$ and $4 - 7 = 9$. The numbers 14 and 2 are both 2 units to the right of a multiple of 12 and -3 and 9 are both 3 units to the left of a multiple of 12. Here we proceed as if $12 = 0$.

Example (Days of the week)

$3 = 10 = 17 = 24$ and $4 + 7 = 4$. Here we proceed as if $7 = 0$.

Periodic Math (Clock Arithmetic)

We will consider as equal all numbers that are in the same relative position between multiples of a fixed positive integer (such as 12).

Example (Time on a clock)

$10 + 4 = 2$ and $4 - 7 = 9$. The numbers 14 and 2 are both 2 units to the right of a multiple of 12 and -3 and 9 are both 3 units to the left of a multiple of 12. Here we proceed as if $12 = 0$.

Example (Days of the week)

$3 = 10 = 17 = 24$ and $4 + 7 = 4$. Here we proceed as if $7 = 0$.

There is nothing special about 12 or 7 in these considerations, from a mathematical point of view. Gauss was the first to recognize the wider significance.

Periodic Math

For a positive integer n (e.g., 7, 12), say $a \equiv b \mod n$ to mean a and b have the *same relative position* between multiples of n (equiv: they have the *same remainder* when divided by n).

For a positive integer n (e.g., 7, 12), say $a \equiv b \bmod n$ to mean a and b have the *same relative position* between multiples of n (equiv: they have the *same remainder* when divided by n).

Example (Integers mod 4)

$9 \equiv 5 \equiv 1 \equiv -3 \bmod 4$ because they are all in the same relative position between multiples of 4:

$$\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$$

They all leave remainder 1 when divided by 4.

For a positive integer n (e.g., 7, 12), say $a \equiv b \bmod n$ to mean a and b have the *same relative position* between multiples of n (equiv: they have the *same remainder* when divided by n).

Example (Integers mod 4)

$9 \equiv 5 \equiv 1 \equiv -3 \bmod 4$ because they are all in the same relative position between multiples of 4:

$$\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$$

They all leave remainder 1 when divided by 4.

When we work "mod n" we essentially ignore n by treating it like 0: $5 \equiv 0 \bmod 5$, $8 \equiv 0 \bmod 8$, $24 \equiv 0 \bmod 8$, and so on.

Adding and Multiplying in Periodic Math

Crucial idea with no clock interpretation: we can multiply too!

Adding and Multiplying in Periodic Math

Crucial idea with no clock interpretation: we can multiply too!

Example

Can we add/multiply $17 \equiv 2 \bmod 5$ and $9 \equiv -1 \bmod 5$?

Adding and Multiplying in Periodic Math

Crucial idea with no clock interpretation: we can multiply too!

Example

Can we add/multiply $17 \equiv 2 \bmod 5$ and $9 \equiv -1 \bmod 5$?

$$17 + 9 \stackrel{?}{\equiv} 2 - 1 \bmod 5 \qquad (26 \stackrel{\checkmark}{\equiv} 1 \bmod 5)$$
$$17 \cdot 9 \stackrel{?}{\equiv} 2 \cdot (-1) \bmod 5 \qquad (153 \stackrel{\checkmark}{\equiv} -2 \bmod 5).$$

Adding and Multiplying in Periodic Math

Crucial idea with no clock interpretation: we can multiply too!

Example

Can we add/multiply $17 \equiv 2 \bmod 5$ and $9 \equiv -1 \bmod 5$?

$$17 + 9 \stackrel{?}{\equiv} 2 - 1 \bmod 5 \qquad (26 \stackrel{\checkmark}{\equiv} 1 \bmod 5)$$

$$17 \cdot 9 \stackrel{?}{\equiv} 2 \cdot (-1) \bmod 5 \qquad (153 \stackrel{\checkmark}{\equiv} -2 \bmod 5).$$

Theorem

If $a \equiv b \bmod m$ and $c \equiv d \bmod m$ then

$$a \pm c \equiv b \pm d \bmod m, \;\; ac \equiv bd \bmod m.$$

Adding and Multiplying in Periodic Math

Crucial idea with no clock interpretation: we can multiply too!

Example

Can we add/multiply $17 \equiv 2 \bmod 5$ and $9 \equiv -1 \bmod 5$?

$$17 + 9 \overset{?}{\equiv} 2 - 1 \bmod 5 \qquad (26 \overset{\checkmark}{\equiv} 1 \bmod 5)$$
$$17 \cdot 9 \overset{?}{\equiv} 2 \cdot (-1) \bmod 5 \qquad (153 \overset{\checkmark}{\equiv} -2 \bmod 5).$$

Theorem

If $a \equiv b \bmod m$ and $c \equiv d \bmod m$ then

$$a \pm c \equiv b \pm d \bmod m, \;\; ac \equiv bd \bmod m.$$

There is generally *no division* by nonzero numbers in modular arithmetic.

Crucial idea with no clock interpretation: we can multiply too!

Example

Can we add/multiply $17 \equiv 2 \bmod 5$ and $9 \equiv -1 \bmod 5$?

$$17 + 9 \stackrel{?}{\equiv} 2 - 1 \bmod 5 \qquad (26 \stackrel{\checkmark}{\equiv} 1 \bmod 5)$$
$$17 \cdot 9 \stackrel{?}{\equiv} 2 \cdot (-1) \bmod 5 \qquad (153 \stackrel{\checkmark}{\equiv} -2 \bmod 5).$$

Theorem

If $a \equiv b \bmod m$ *and* $c \equiv d \bmod m$ *then*

$$a \pm c \equiv b \pm d \bmod m, \quad ac \equiv bd \bmod m.$$

There is generally *no division* by nonzero numbers in modular arithmetic. For example, $3 \not\equiv 8 \bmod 10$ but upon multiplying both sides by 2 we have $6 \equiv 16 \bmod 10$ and we cannot "cancel the 2's".

Adding and Multiplying in Periodic Math

Crucial idea with no clock interpretation: we can multiply too!

Example

Can we add/multiply $17 \equiv 2 \bmod 5$ and $9 \equiv -1 \bmod 5$?

$$17 + 9 \stackrel{?}{\equiv} 2 - 1 \bmod 5 \qquad (26 \stackrel{\checkmark}{\equiv} 1 \bmod 5)$$
$$17 \cdot 9 \stackrel{?}{\equiv} 2 \cdot (-1) \bmod 5 \qquad (153 \stackrel{\checkmark}{\equiv} -2 \bmod 5).$$

Theorem

If $a \equiv b \bmod m$ *and* $c \equiv d \bmod m$ *then*

$$a \pm c \equiv b \pm d \bmod m, \quad ac \equiv bd \bmod m.$$

There is generally *no division* by nonzero numbers in modular arithmetic. For example, $3 \not\equiv 8 \bmod 10$ but upon multiplying both sides by 2 we have $6 \equiv 16 \bmod 10$ and we cannot "cancel the 2's".
Let's work on Exercise 4, applying modular arithmetic.

Powers in Periodic Math

The key to period lengths of decimals is *powers* (of 10) mod n.

Example

Successive powers of 2 mod 7 are $2, 4, 1, 2, 4, 1, \ldots$.

The key to period lengths of decimals is *powers* (of 10) mod n.

Example

Successive powers of 2 mod 7 are $2, 4, 1, 2, 4, 1, \ldots$.

What are the powers of 10 mod 11? Or 10 mod 27?

Powers in Periodic Math

The key to period lengths of decimals is *powers* (of 10) mod n.

Example

Successive powers of 2 mod 7 are $2, 4, 1, 2, 4, 1, \ldots$.

What are the powers of 10 mod 11? Or 10 mod 27?
Working mod 11, there are only two powers of 10:

$$10^1 \equiv 10 \bmod 11,$$

The key to period lengths of decimals is *powers* (of 10) mod n.

Example

Successive powers of 2 mod 7 are $2, 4, 1, 2, 4, 1, \ldots$.

What are the powers of 10 mod 11? Or 10 mod 27?
Working mod 11, there are only two powers of 10:

$$\begin{aligned} 10^1 &\equiv 10 \bmod 11, \\ 10^2 = 100 &\equiv 1 \bmod 11, \end{aligned}$$

Powers in Periodic Math

The key to period lengths of decimals is *powers* (of 10) mod n.

Example

Successive powers of 2 mod 7 are $2, 4, 1, 2, 4, 1, \ldots$.

What are the powers of 10 mod 11? Or 10 mod 27?
Working mod 11, there are only two powers of 10:

$$\begin{aligned} 10^1 &\equiv 10 \bmod 11, \\ 10^2 = 100 &\equiv 1 \bmod 11, \\ 10^3 = 10 \cdot 10^2 &\equiv 10 \bmod 11, \end{aligned}$$

Powers in Periodic Math

The key to period lengths of decimals is *powers* (of 10) mod n.

Example

Successive powers of 2 mod 7 are $2, 4, 1, 2, 4, 1, \ldots$.

What are the powers of 10 mod 11? Or 10 mod 27?
Working mod 11, there are only two powers of 10:

$$\begin{aligned} 10^1 &\equiv 10 \bmod 11, \\ 10^2 = 100 &\equiv 1 \bmod 11, \\ 10^3 = 10 \cdot 10^2 &\equiv 10 \bmod 11, \\ 10^4 = 10 \cdot 10^3 &\equiv 1 \bmod 11, \text{ and so on } (10, 1, 10, 1...). \end{aligned}$$

Let's work out powers of 10 mod 27.

The key to period lengths of decimals is *powers* (of 10) mod n.

Example

Successive powers of 2 mod 7 are $2, 4, 1, 2, 4, 1, \ldots$.

What are the powers of 10 mod 11? Or 10 mod 27?
Working mod 11, there are only two powers of 10:

$$
\begin{aligned}
10^1 &\equiv 10 \bmod 11, \\
10^2 = 100 &\equiv 1 \bmod 11, \\
10^3 = 10 \cdot 10^2 &\equiv 10 \bmod 11, \\
10^4 = 10 \cdot 10^3 &\equiv 1 \bmod 11, \text{ and so on (10, 1, 10, 1...).}
\end{aligned}
$$

Let's work out powers of 10 mod 27.

$$10^1 \equiv 10 \bmod 27,$$

Powers in Periodic Math

The key to period lengths of decimals is *powers* (of 10) mod n.

Example

Successive powers of 2 mod 7 are $2, 4, 1, 2, 4, 1, \ldots$.

What are the powers of 10 mod 11? Or 10 mod 27?
Working mod 11, there are only two powers of 10:

$$
\begin{aligned}
10^1 &\equiv 10 \bmod 11, \\
10^2 = 100 &\equiv 1 \bmod 11, \\
10^3 = 10 \cdot 10^2 &\equiv 10 \bmod 11, \\
10^4 = 10 \cdot 10^3 &\equiv 1 \bmod 11, \text{ and so on } (10,\ 1,\ 10,\ 1...).
\end{aligned}
$$

Let's work out powers of 10 mod 27.

$$
\begin{aligned}
10^1 &\equiv 10 \bmod 27, \\
10^2 = 100 &\equiv 19 \bmod 27,
\end{aligned}
$$

The key to period lengths of decimals is *powers* (of 10) mod n.

Example

Successive powers of 2 mod 7 are $2, 4, 1, 2, 4, 1, \ldots$.

What are the powers of 10 mod 11? Or 10 mod 27?
Working mod 11, there are only two powers of 10:

$$\begin{aligned} 10^1 &\equiv 10 \bmod 11, \\ 10^2 = 100 &\equiv 1 \bmod 11, \\ 10^3 = 10 \cdot 10^2 &\equiv 10 \bmod 11, \\ 10^4 = 10 \cdot 10^3 &\equiv 1 \bmod 11, \text{ and so on (10, 1, 10, 1...).} \end{aligned}$$

Let's work out powers of 10 mod 27.

$$\begin{aligned} 10^1 &\equiv 10 \bmod 27, \\ 10^2 = 100 &\equiv 19 \bmod 27, \\ 10^3 = 10 \cdot 10^2 &\equiv 1 \bmod 27, \end{aligned}$$

Powers in Periodic Math

The key to period lengths of decimals is *powers* (of 10) mod n.

Example

Successive powers of 2 mod 7 are $2, 4, 1, 2, 4, 1, \ldots$.

What are the powers of 10 mod 11? Or 10 mod 27?
Working mod 11, there are only two powers of 10:

$$\begin{aligned} 10^1 &\equiv 10 \bmod 11, \\ 10^2 = 100 &\equiv 1 \bmod 11, \\ 10^3 = 10 \cdot 10^2 &\equiv 10 \bmod 11, \\ 10^4 = 10 \cdot 10^3 &\equiv 1 \bmod 11, \text{ and so on } (10, 1, 10, 1...). \end{aligned}$$

Let's work out powers of 10 mod 27.

$$\begin{aligned} 10^1 &\equiv 10 \bmod 27, \\ 10^2 = 100 &\equiv 19 \bmod 27, \\ 10^3 = 10 \cdot 10^2 &\equiv 1 \bmod 27, \\ 10^4 = 10 \cdot 10^3 &\equiv 10 \bmod 27, \end{aligned}$$

Powers in Periodic Math

The key to period lengths of decimals is *powers* (of 10) mod n.

Example

Successive powers of 2 mod 7 are $2, 4, 1, 2, 4, 1, \ldots$.

What are the powers of 10 mod 11? Or 10 mod 27?
Working mod 11, there are only two powers of 10:

$$\begin{aligned} 10^1 &\equiv 10 \bmod 11, \\ 10^2 = 100 &\equiv 1 \bmod 11, \\ 10^3 = 10 \cdot 10^2 &\equiv 10 \bmod 11, \\ 10^4 = 10 \cdot 10^3 &\equiv 1 \bmod 11, \text{ and so on (10, 1, 10, 1...).} \end{aligned}$$

Let's work out powers of 10 mod 27.

$$\begin{aligned} 10^1 &\equiv 10 \bmod 27, \\ 10^2 = 100 &\equiv 19 \bmod 27, \\ 10^3 = 10 \cdot 10^2 &\equiv 1 \bmod 27, \\ 10^4 = 10 \cdot 10^3 &\equiv 10 \bmod 27, \text{ etc. (19, 10, 1, 19, 10, 1,...).} \end{aligned}$$

Powers of 10 and Period Lengths

k	1	2	3	4	5	6	7	8	9	10
10^k mod 3	1	1	1	1	1	1	1	1	1	1
10^k mod 7	3	2	6	4	5	1	3	2	6	4
10^k mod 9	1	1	1	1	1	1	1	1	1	1
10^k mod 11	10	1	10	1	10	1	10	1	10	1
10^k mod 13	10	9	12	3	4	1	10	9	12	3
10^k mod 21	10	16	13	4	19	1	10	16	13	4
10^k mod 27	10	19	1	10	19	1	10	19	1	10
10^k mod 41	10	18	16	37	1	10	18	16	37	1

Once we know $10^3 \equiv 16 \bmod 41$, to find $10^4 \bmod 41$ *don't start from scratch*.

Powers of 10 and Period Lengths

k	1	2	3	4	5	6	7	8	9	10
10^k mod 3	1	1	1	1	1	1	1	1	1	1
10^k mod 7	3	2	6	4	5	1	3	2	6	4
10^k mod 9	1	1	1	1	1	1	1	1	1	1
10^k mod 11	10	1	10	1	10	1	10	1	10	1
10^k mod 13	10	9	12	3	4	1	10	9	12	3
10^k mod 21	10	16	13	4	19	1	10	16	13	4
10^k mod 27	10	19	1	10	19	1	10	19	1	10
10^k mod 41	10	18	16	37	1	10	18	16	37	1

Once we know $10^3 \equiv 16 \bmod 41$, to find $10^4 \bmod 41$ *don't start from scratch*. Instead: $10^4 = 10 \cdot 10^3 \equiv 10 \cdot 16 \bmod 41$, and $160 \equiv 37 \bmod 41$.

Powers of 10 and Period Lengths

k	1	2	3	4	5	6	7	8	9	10
10^k mod 3	1	1	1	1	1	1	1	1	1	1
10^k mod 7	3	2	6	4	5	1	3	2	6	4
10^k mod 9	1	1	1	1	1	1	1	1	1	1
10^k mod 11	10	1	10	1	10	1	10	1	10	1
10^k mod 13	10	9	12	3	4	1	10	9	12	3
10^k mod 21	10	16	13	4	19	1	10	16	13	4
10^k mod 27	10	19	1	10	19	1	10	19	1	10
10^k mod 41	10	18	16	37	1	10	18	16	37	1

Once we know $10^3 \equiv 16 \bmod 41$, to find $10^4 \bmod 41$ *don't start from scratch*. Instead: $10^4 = 10 \cdot 10^3 \equiv 10 \cdot 16 \bmod 41$, and $160 \equiv 37 \bmod 41$. **So can compute** $10^k \bmod 13$ **by bare hands!**

Powers of 10 and Period Lengths

k	1	2	3	4	5	6	7	8	9	10
10^k mod 3	1	1	1	1	1	1	1	1	1	1
10^k mod 7	3	2	6	4	5	1	3	2	6	4
10^k mod 9	1	1	1	1	1	1	1	1	1	1
10^k mod 11	10	1	10	1	10	1	10	1	10	1
10^k mod 13	10	9	12	3	4	1	10	9	12	3
10^k mod 21	10	16	13	4	19	1	10	16	13	4
10^k mod 27	10	19	1	10	19	1	10	19	1	10
10^k mod 41	10	18	16	37	1	10	18	16	37	1

Once we know $10^3 \equiv 16 \bmod 41$, to find $10^4 \bmod 41$ *don't start from scratch*. Instead: $10^4 = 10 \cdot 10^3 \equiv 10 \cdot 16 \bmod 41$, and $160 \equiv 37 \bmod 41$. **So can compute** $10^k \bmod 13$ **by bare hands!**

Theorem (Gauss' formula)

For n not divisible by 2 or 5, $L(n)$ is the number of different powers of 10 mod n.

A “Formula” for $L(n)$, the Decimal Period Length of $1/n$

Where does Gauss' Formula come from? For n not divisible by 2 or 5, we've seen:

$$L(n) \quad = \quad \text{smallest } d > 0 \text{ such that } 10^d - 1 \text{ is a multiple of } n$$

A "Formula" for $L(n)$, the Decimal Period Length of $1/n$

Where does Gauss' Formula come from? For n not divisible by 2 or 5, we've seen:

$$\begin{aligned} L(n) &= \text{smallest } d > 0 \text{ such that } 10^d - 1 \text{ is a multiple of } n \\ &= \text{smallest } d > 0 \text{ such that } 10^d \equiv 1 \bmod n, \end{aligned}$$

so $10^k \bmod n$ cycles for $k > d$ (**and cycle ends at 1; why?**).

A "Formula" for $L(n)$, the Decimal Period Length of $1/n$

Where does Gauss' Formula come from? For n not divisible by 2 or 5, we've seen:

$$\begin{aligned} L(n) &= \text{smallest } d > 0 \text{ such that } 10^d - 1 \text{ is a multiple of } n \\ &= \text{smallest } d > 0 \text{ such that } 10^d \equiv 1 \bmod n, \end{aligned}$$

so $10^k \bmod n$ cycles for $k > d$ (**and cycle ends at 1; why?**). We've also seen that n has a multiple $10^d - 1$ with $d \leq n - 1$, so

$$L(n) \leq n - 1.$$

$$\begin{aligned} 10 - 1 &= 3^2 \\ 10^2 - 1 &= 3^2 \cdot 11 \\ 10^3 - 1 &= 3^3 \cdot 37 \\ 10^4 - 1 &= 3^2 \cdot 11 \cdot 101 \\ 10^5 - 1 &= 3^2 \cdot 41 \cdot 271 \\ 10^6 - 1 &= 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \end{aligned}$$

A "Formula" for $L(n)$, the Decimal Period Length of $1/n$

Where does Gauss' Formula come from? For n not divisible by 2 or 5, we've seen:

$$\begin{aligned} L(n) &= \text{smallest } d > 0 \text{ such that } 10^d - 1 \text{ is a multiple of } n \\ &= \text{smallest } d > 0 \text{ such that } 10^d \equiv 1 \bmod n, \end{aligned}$$

so $10^k \bmod n$ cycles for $k > d$ (**and cycle ends at 1; why?**). We've also seen that n has a multiple $10^d - 1$ with $d \leq n - 1$, so

$$L(n) \leq n - 1.$$

$$\begin{aligned} 10 - 1 &= 3^2 \\ 10^2 - 1 &= 3^2 \cdot 11 \\ 10^3 - 1 &= 3^3 \cdot 37 \\ 10^4 - 1 &= 3^2 \cdot 11 \cdot 101 \\ 10^5 - 1 &= 3^2 \cdot 41 \cdot 271 \\ 10^6 - 1 &= 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \end{aligned}$$

So $\frac{1}{7}$ has decimal period length 6 and $\frac{1}{37}$ **has period length**

A "Formula" for $L(n)$, the Decimal Period Length of $1/n$

Where does Gauss' Formula come from? For n not divisible by 2 or 5, we've seen:

$$\begin{aligned} L(n) &= \text{smallest } d > 0 \text{ such that } 10^d - 1 \text{ is a multiple of } n \\ &= \text{smallest } d > 0 \text{ such that } 10^d \equiv 1 \bmod n, \end{aligned}$$

so $10^k \bmod n$ cycles for $k > d$ (**and cycle ends at 1; why?**).
We've also seen that n has a multiple $10^d - 1$ with $d \leq n - 1$, so

$$L(n) \leq n - 1.$$

$$\begin{aligned} 10 - 1 &= 3^2 \\ 10^2 - 1 &= 3^2 \cdot 11 \\ 10^3 - 1 &= 3^3 \cdot 37 \\ 10^4 - 1 &= 3^2 \cdot 11 \cdot 101 \\ 10^5 - 1 &= 3^2 \cdot 41 \cdot 271 \\ 10^6 - 1 &= 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \end{aligned}$$

So $\frac{1}{7}$ has decimal period length 6 and $\frac{1}{37}$ **has period length** 3.

Cyclic Shifts for Denominator 7

Considering $10^k \bmod n$ further will help us to understand the cycling pattern in periodic parts. For $n = 7$, first arrange the decimals in *successive cyclic order* (i.e., shift to the right):

Fraction	Decimal	Fraction	Decimal
1/7	$.\overline{142857}$	1/7	$.\overline{142857}$
2/7	$.\overline{285714}$	5/7	$.\overline{714285}$
3/7	$.\overline{428571}$	4/7	$.\overline{571428}$
4/7	$.\overline{571428}$	6/7	$.\overline{857142}$
5/7	$.\overline{714285}$	2/7	$.\overline{285714}$
6/7	$.\overline{857142}$	3/7	$.\overline{428571}$

Cyclic Shifts for Denominator 7

Considering 10^k mod n further will help us to understand the cycling pattern in periodic parts. For $n = 7$, first arrange the decimals in *successive cyclic order* (i.e., shift to the right):

Fraction	Decimal	Fraction	Decimal
$1/7$	$.\overline{142857}$	$1/7$	$.\overline{142857}$
$2/7$	$.\overline{285714}$	$5/7$	$.\overline{714285}$
$3/7$	$.\overline{428571}$	$4/7$	$.\overline{571428}$
$4/7$	$.\overline{571428}$	$6/7$	$.\overline{857142}$
$5/7$	$.\overline{714285}$	$2/7$	$.\overline{285714}$
$6/7$	$.\overline{857142}$	$3/7$	$.\overline{428571}$

Cyclic shifts in the *other* order (i.e., to the left) will turn out to be more useful because that matches a natural algebraic way of shifting decimals: multiply by 10 (e.g., $10 \cdot (.\overline{142857}) = 1.\overline{428571}$).

Cyclic Shifts for Denominator 7

Considering 10^k mod n further will help us to understand the cycling pattern in periodic parts. For $n = 7$, first arrange the decimals in *successive cyclic order* (i.e., shift to the right):

Fraction	Decimal	Fraction	Decimal
$1/7$	$.\overline{142857}$	$1/7$	$.\overline{142857}$
$2/7$	$.\overline{285714}$	$5/7$	$.\overline{714285}$
$3/7$	$.\overline{428571}$	$4/7$	$.\overline{571428}$
$4/7$	$.\overline{571428}$	$6/7$	$.\overline{857142}$
$5/7$	$.\overline{714285}$	$2/7$	$.\overline{285714}$
$6/7$	$.\overline{857142}$	$3/7$	$.\overline{428571}$

Cyclic shifts in the *other* order (i.e., to the left) will turn out to be more useful because that matches a natural algebraic way of shifting decimals: multiply by 10 (e.g., $10 \cdot (.\overline{142857}) = 1.\overline{428571}$).

The decimal moves to the **right** when multiplying by 10, so *relative to the decimal point* the sequence of digits moves **left**.

Cyclic Shifts for Denominator 7

Cyclic shifts in the other order are better because it matches a natural algebraic way of shifting decimals: multiply by 10.

Fraction	Decimal	Fraction	Decimal
1/7	$.\overline{142857}$	1/7	$.\overline{142857}$
2/7	$.\overline{285714}$	3/7	$.\overline{428571}$
3/7	$.\overline{428571}$	2/7	$.\overline{285714}$
4/7	$.\overline{571428}$	6/7	$.\overline{857142}$
5/7	$.\overline{714285}$	4/7	$.\overline{571428}$
6/7	$.\overline{857142}$	5/7	$.\overline{714285}$

Cyclic Shifts for Denominator 7

Cyclic shifts in the other order are better because it matches a natural algebraic way of shifting decimals: multiply by 10.

Fraction	Decimal	Fraction	Decimal
1/7	$.\overline{142857}$	1/7	$.\overline{142857}$
2/7	$.\overline{285714}$	3/7	$.\overline{428571}$
3/7	$.\overline{428571}$	2/7	$.\overline{285714}$
4/7	$.\overline{571428}$	6/7	$.\overline{857142}$
5/7	$.\overline{714285}$	4/7	$.\overline{571428}$
6/7	$.\overline{857142}$	5/7	$.\overline{714285}$

The new ordered sequence of *numerators* $1, 3, 2, 6, 4, 5$ on the right has a nice interpretation. **Any guesses?** (Hint: Think mod 7.)

Cyclic Shifts for Denominator 7

Cyclic shifts in the other order are better because it matches a natural algebraic way of shifting decimals: multiply by 10.

Fraction	Decimal	Fraction	Decimal
$1/7$	$.\overline{142857}$	$1/7$	$.\overline{142857}$
$2/7$	$.\overline{285714}$	$3/7$	$.\overline{428571}$
$3/7$	$.\overline{428571}$	$2/7$	$.\overline{285714}$
$4/7$	$.\overline{571428}$	$6/7$	$.\overline{857142}$
$5/7$	$.\overline{714285}$	$4/7$	$.\overline{571428}$
$6/7$	$.\overline{857142}$	$5/7$	$.\overline{714285}$

The new ordered sequence of *numerators* $1, 3, 2, 6, 4, 5$ on the right has a nice interpretation. **Any guesses?** (Hint: Think mod 7.) These are the *remainders* when 10^k is divided by 7; in other words, the corresponding reduced proper fractions $a/7$ are (in order) the **fractional parts** of $10^k/7$ ($k = 0, 1, 2, 3, \ldots$).

Cyclic Shifts for Denominator 7

k	0	1	2	3	4	5	6	7	8	9	10
$10^k \bmod 7$	1	3	2	6	4	5	1	3	2	6	4

Fraction	Decimal	Fraction	Decimal
$1/7$	$.\overline{142857}$	$1/7$	$.\overline{142857}$
$10/7$	$1.\overline{428571}$	$3/7$	$.\overline{428571}$
$10^2/7$	$14.\overline{285714}$	$2/7$	$.\overline{285714}$
$10^3/7$	$142.\overline{857142}$	$6/7$	$.\overline{857142}$
$10^4/7$	$1428.\overline{571428}$	$4/7$	$.\overline{571428}$
$10^5/7$	$14285.\overline{714285}$	$5/7$	$.\overline{714285}$

Cyclic Shifts for Denominator 13: 2 Cyclic Shifts

The powers 10^k mod 13 are $1, 10, 9, 12, 3, 4, 1, 10, 9, \ldots$

Fraction	Decimal	Fraction	Decimal
1/13	$.\overline{076923}$	2/13	$.\overline{153846}$
10/13	$.\overline{769230}$	7/13	$.\overline{538461}$
9/13	$.\overline{692307}$	5/13	$.\overline{384615}$
12/13	$.\overline{923076}$	11/13	$.\overline{846153}$
3/13	$.\overline{230769}$	6/13	$.\overline{461538}$
4/13	$.\overline{307692}$	8/13	$.\overline{615384}$

Cyclic Shifts for Denominator 13: 2 Cyclic Shifts

The powers 10^k mod 13 are $1, 10, 9, 12, 3, 4, 1, 10, 9, \ldots$

Fraction	Decimal	Fraction	Decimal
1/13	$.\overline{076923}$	2/13	$.\overline{153846}$
10/13	$.\overline{769230}$	7/13	$.\overline{538461}$
9/13	$.\overline{692307}$	5/13	$.\overline{384615}$
12/13	$.\overline{923076}$	11/13	$.\overline{846153}$
3/13	$.\overline{230769}$	6/13	$.\overline{461538}$
4/13	$.\overline{307692}$	8/13	$.\overline{615384}$

We missed 2, and $10^k \cdot 2$ mod 13 cycles the rest: 2, 7, 5, 11, 6, 8.

Example

$10 \cdot 2 = 20 \equiv 7$ mod 13, $10^2 \cdot 2 \equiv 5$ mod 13, $10^3 \cdot 2 \equiv 11$ mod 13

Cyclic Shifts for Denominator 13: 2 Cyclic Shifts

The powers 10^k mod 13 are $1, 10, 9, 12, 3, 4, 1, 10, 9, \ldots$

Fraction	Decimal	Fraction	Decimal
$1/13$	$.\overline{076923}$	$2/13$	$.\overline{153846}$
$10/13$	$.\overline{769230}$	$7/13$	$.\overline{538461}$
$9/13$	$.\overline{692307}$	$5/13$	$.\overline{384615}$
$12/13$	$.\overline{923076}$	$11/13$	$.\overline{846153}$
$3/13$	$.\overline{230769}$	$6/13$	$.\overline{461538}$
$4/13$	$.\overline{307692}$	$8/13$	$.\overline{615384}$

We missed 2, and $10^k \cdot 2$ mod 13 cycles the rest: 2, 7, 5, 11, 6, 8.

Example

$10 \cdot 2 = 20 \equiv 7 \bmod 13$, $10^2 \cdot 2 \equiv 5 \bmod 13$, $10^3 \cdot 2 \equiv 11 \bmod 13$

Theorem

If n not div. by 2 or 5, $\frac{a}{n}$ and $\frac{b}{n}$ have same decimal cycle if and only if $a \equiv 10^k b \bmod n$.

Cyclic Shifts and decimal cycles

Theorem

If n not div. by 2 or 5, $\frac{a}{n}$ and $\frac{b}{n}$ have same decimal cycle if and only if $a \equiv 10^k b \mod n$.

Why is this true? Think about each direction of implication separately.

Cyclic Shifts and decimal cycles

Theorem

If n not div. by 2 or 5, $\frac{a}{n}$ and $\frac{b}{n}$ have same decimal cycle if and only if $a \equiv 10^k b \bmod n$.

Why is this true? Think about each direction of implication separately. This will unlock the mystery of Gauss' Theorem:

Corollary

If n is not divisible by 2 or 5 then

$$L(n) \cdot \#\text{cycles} = \#\text{reduced proper fractions}$$

Cyclic Shifts and decimal cycles

Theorem

If n not div. by 2 or 5, $\frac{a}{n}$ and $\frac{b}{n}$ have same decimal cycle if and only if $a \equiv 10^k b \bmod n$.

Why is this true? Think about each direction of implication separately. This will unlock the mystery of Gauss' Theorem:

Corollary

If n is not divisible by 2 or 5 then

$$L(n) \cdot \#\text{cycles} = \#\text{reduced proper fractions}$$

To understand this, break up the collection of reduced proper fractions a/n into "orbits" under repeated multiplications by 10.
Let's try to conclude using two observations:

1. Each "orbit" is exactly one cycle of a decimal period. (Why?)
2. We saw that all cycles have the *same* length, namely $L(n)$.

Maximal Period Length for $1/n$ and an Unsolved Conjecture

For every n not divisible by 2 or 5, $L(n) \leq n - 1$. When does $L(n) = n - 1$?

Maximal Period Length for $1/n$ and an Unsolved Conjecture

For every n not divisible by 2 or 5, $L(n) \leq n-1$. When does $L(n) = n-1$? By the *proof* via orbits under multiplication by 10^k mod n that

$$L(n) \cdot \#\text{cycles} = \#\text{reduced proper fractions} \leq n-1,$$

we see $L(n) = n-1$ precisely when 10^k mod n sweeps out *all* non-zero remainders modulo n. (**Why?**)

Maximal Period Length for $1/n$ and an Unsolved Conjecture

For every n not divisible by 2 or 5, $L(n) \leq n - 1$. When does $L(n) = n - 1$? By the *proof* via orbits under multiplication by 10^k mod n that

$$L(n) \cdot \#\text{cycles} = \#\text{reduced proper fractions} \leq n - 1,$$

we see $L(n) = n - 1$ precisely when 10^k mod n sweeps out *all* non-zero remainders modulo n. (**Why?**) If $n < 100$, it occurs for

$$7, \ 17, \ 19, \ 23, \ 29, \ 47, \ 59, \ 61, \ 97.$$

Those specific n are prime. **Is this a coincidence?**

Maximal Period Length for $1/n$ and an Unsolved Conjecture

For every n not divisible by 2 or 5, $L(n) \leq n-1$. When does $L(n) = n-1$? By the *proof* via orbits under multiplication by 10^k mod n that

$$L(n) \cdot \#\text{cycles} = \#\text{reduced proper fractions} \leq n-1,$$

we see $L(n) = n-1$ precisely when 10^k mod n sweeps out *all* non-zero remainders modulo n. (**Why?**) If $n < 100$, it occurs for

$$7,\ 17,\ 19,\ 23,\ 29,\ 47,\ 59,\ 61,\ 97.$$

Those specific n are prime. **Is this a coincidence?** No. There are $n-1$ reduced proper fractions with denominator n precisely for prime n (**why?**), so:

Corollary

If $L(n) = n-1$ then n is prime.

Maximal Period Length for $1/n$ and an Unsolved Conjecture

For every n not divisible by 2 or 5, $L(n) \leq n - 1$. When does $L(n) = n - 1$? By the *proof* via orbits under multiplication by 10^k mod n that

$$L(n) \cdot \#\text{cycles} = \#\text{reduced proper fractions} \leq n - 1,$$

we see $L(n) = n - 1$ precisely when 10^k mod n sweeps out *all* non-zero remainders modulo n. (**Why?**) If $n < 100$, it occurs for

$$7,\ 17,\ 19,\ 23,\ 29,\ 47,\ 59,\ 61,\ 97.$$

Those specific n are prime. **Is this a coincidence?** No. There are $n - 1$ reduced proper fractions with denominator n precisely for prime n (**why?**), so:

Corollary

If $L(n) = n - 1$ then n is prime.

The converse statement is false at primes $3, 11, 13, 31, 37, 41, \ldots$
There is no known "formula" telling us **exactly** *when $L(p) = p - 1$.*

Maximal Period Length for $1/n$ and an Unsolved Conjecture

For prime $p \neq 2, 5$, when does $L(p) = p - 1$?

Conjecture

There are infinitely many primes p such that $1/p$ *has "full" decimal period length* $p - 1$ *(*$\approx .3739558 \sim 37.4\%$ *of all p).*

Maximal Period Length for $1/n$ and an Unsolved Conjecture

For prime $p \neq 2, 5$, when does $L(p) = p - 1$?

Conjecture

There are infinitely many primes p such that $1/p$ has "full" decimal period length $p - 1$ ($\approx .3739558 \sim 37.4\%$ of all p).

But why focus only on base 10? **Is 10 special in math?**

For prime $p \neq 2, 5$, when does $L(p) = p - 1$?

Conjecture

There are infinitely many primes p such that $1/p$ has "full" decimal period length $p - 1$ ($\approx .3739558 \sim 37.4\%$ of all p).

But why focus only on base 10? **Is 10 special in math?** No! There is a more general conjecture, due to Emil Artin:

Conjecture

For any non-square $b \geq 2$, there are infinitely many primes p such that $1/p$ has base-b "decimal" expansion with period length $p - 1$.

Maximal Period Length for $1/n$ and an Unsolved Conjecture

For prime $p \neq 2, 5$, when does $L(p) = p - 1$?

Conjecture

There are infinitely many primes p such that $1/p$ has "full" decimal period length $p - 1$ ($\approx .3739558 \sim 37.4\%$ of all p).

But why focus only on base 10? **Is 10 special in math?** No! There is a more general conjecture, due to Emil Artin:

Conjecture

For any non-square $b \geq 2$, there are infinitely many primes p such that $1/p$ has base-b "decimal" expansion with period length $p - 1$.

For $b = 10$ this is first conjecture above. The proportion of p in Artin's conjecture is predicted exactly (depending on b).

Maximal Period Length for $1/n$ and an Unsolved Conjecture

For prime $p \neq 2, 5$, when does $L(p) = p - 1$?

Conjecture

There are infinitely many primes p such that $1/p$ has "full" decimal period length $p - 1$ ($\approx .3739558 \sim 37.4\%$ of all p).

But why focus only on base 10? **Is 10 special in math?** No! There is a more general conjecture, due to Emil Artin:

Conjecture

For any non-square $b \geq 2$, there are infinitely many primes p such that $1/p$ has base-b "decimal" expansion with period length $p - 1$.

For $b = 10$ this is first conjecture above. The proportion of p in Artin's conjecture is predicted exactly (depending on b).

Gauss' insights will enable us to reformulate this conjecture in a more illuminating manner...

Conjecture

For any non-square $b \geq 2$, there are infinitely many primes p such that $1/p$ has base-b "decimal" expansion with period length $p-1$.

By Gauss' Theorem, for $b = 10$ the condition of period length $p-1$ (with $p \neq 2, 5$) is *exactly* the same as there being a single cycle for the periodic parts. **Why?**

Conjecture

For any non-square $b \geq 2$, there are infinitely many primes p such that $1/p$ has base-b "decimal" expansion with period length $p - 1$.

By Gauss' Theorem, for $b = 10$ the condition of period length $p - 1$ (with $p \neq 2, 5$) is *exactly* the same as there being a single cycle for the periodic parts. **Why?** It is equivalent that $10^k \bmod p$ sweeps through *all* non-zero remainders mod p. **Why?**

Maximal Period Length for $1/n$ and an Unsolved Conjecture

Conjecture

For any non-square $b \geq 2$, there are infinitely many primes p such that $1/p$ has base-b "decimal" expansion with period length $p-1$.

By Gauss' Theorem, for $b = 10$ the condition of period length $p-1$ (with $p \neq 2, 5$) is *exactly* the same as there being a single cycle for the periodic parts. **Why?** It is equivalent that $10^k \bmod p$ sweeps through *all* non-zero remainders mod p. **Why?**

Gauss' reasoning works for general b, so Artin's conjecture says:

Conjecture

For any non-square $b \geq 2$, there are infinitely many primes p such that $b^k \bmod p$ sweeps through all non-zero remainders mod p.

Conjecture

For any non-square $b \geq 2$, there are infinitely many primes p such that $1/p$ has base-b "decimal" expansion with period length $p - 1$.

By Gauss' Theorem, for $b = 10$ the condition of period length $p - 1$ (with $p \neq 2, 5$) is *exactly* the same as there being a single cycle for the periodic parts. **Why?** It is equivalent that $10^k \bmod p$ sweeps through *all* non-zero remainders mod p. **Why?**

Gauss' reasoning works for general b, so Artin's conjecture says:

Conjecture

For any non-square $b \geq 2$, there are infinitely many primes p such that $b^k \bmod p$ sweeps through all non-zero remainders mod p.

For square b one can show $b^{(p-1)/2} \equiv 1 \bmod p$ (due to Fermat), so at most $(p-1)/2$ possibilities for $b^k \bmod p$. This is why b is a non-square in Artin's conjecture (which also permits any $b < -1$).

Conjecture

For any non-square $b \geq 2$, there are infinitely many primes p such that $1/p$ has base-b "decimal" expansion with period length $p - 1$.

By Gauss' Theorem, for $b = 10$ the condition of period length $p - 1$ (with $p \neq 2, 5$) is *exactly* the same as there being a single cycle for the periodic parts. **Why?** It is equivalent that $10^k \bmod p$ sweeps through *all* non-zero remainders mod p. **Why?**

Gauss' reasoning works for general b, so Artin's conjecture says:

Conjecture

For any non-square $b \geq 2$, there are infinitely many primes p such that $b^k \bmod p$ sweeps through all non-zero remainders mod p.

For square b one can show $b^{(p-1)/2} \equiv 1 \bmod p$ (due to Fermat), so at most $(p - 1)/2$ possibilities for $b^k \bmod p$. This is why b is a non-square in Artin's conjecture (which also permits any $b < -1$).

The Generalized Riemann Hypothesis implies Artin's conjecture!

References

- M. Bullynck, *Decimal Periods and their Tables: A Research Topic* (1765–1801), Historia Mathematica **36** (2009), 137–160.
- L. E. Dickson, "History of the Theory of Numbers," Vol. I. Chelsea, 1981. (Chapter VI)
- C. F. Gauss, "Disquisitiones Arithmeticae," Yale Univ. Press, 1966. (Art. 312–318)
- W. LeVeque, "Fundamentals of Number Theory", Dover, 1996. (§3.1–3.3, rests on Chapter 2)
- H. Rademacher and O. Toeplitz, "The Enjoyment of Mathematics," Dover, 1990. (Chapter 23)