



## “PROOF BY CRAYON” TILING EXPLORATIONS

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**Overview.** The goal of this activity is to help students discover and use parity to explain why some tiling problems are impossible. Along the way they will engage in research at an elementary level by forming and testing conjectures and asking original questions.

**Materials.** A supply of graph paper is indispensable. I would avoid a worksheet for this activity, since a plain sheet of graph paper will encourage more original thinking.

**Presentation.** The entire exploration can last from 20 to 40 minutes, depending.

a. I like to begin with extremely straight-forward examples. So draw a  $4 \times 4$  grid on the board and write above it, “Tile this  $4 \times 4$  grid with  $1 \times 2$  dominoes.” I also like to find out what the kids already know, so I usually ask, “What does it mean to tile a board with dominoes?” Typical responses usual go along the lines of “Cover the board with dominoes!” The appropriate response, of course, is to cover it with wildly overlapping dominoes. Continue in this manner until the class arrives at a precise definition, such as, “Completely cover the board with dominoes which do not overlap or extend past the edge of the board.”

b. Accomplishing the above tiling is routine, which is a great chance to allow a student who doesn’t come up with the answer very often to present their solution at the board. Now make it more interesting by erasing one of the corner squares and asking students to tile the resulting figure. Eventually the class should decide that this is impossible, but make sure they manage to clearly state why before proceeding. (Since each domino covers two squares, a collection of dominos will always cover an even number of squares. But our figure has fifteen squares, so a tiling is not possible.)

c. Now suggest that we should remove two squares to make a tiling that is possible again. Ask how many dominoes will be required. (Seven.) Have students draw a  $4 \times 4$  board, shade in two squares, and then attempt to tile what remains. Demo this as you go with your figure on the board, making sure to shade in two *adjacent* squares; tiling the rest should then be almost automatic. The surprising fact is that some students will have no trouble, but others will get stuck repeatedly and eventually give up, claiming that “This is impossible!” Suggest that perhaps they are not trying hard enough. But not for too long—explain that their diagram may well be impossible, but claiming that this is the case is a tall order and needs to be backed up with a solid explanation.

d. A few things could happen at this point. First of all, it might be a good thing to check whether or not any students are familiar with this activity. (It is a classic and well-known problem, in certain circles.) If so, ask this student to refrain from shouting out answers by promising to let them explain the main idea at a later point. Second, students may feel that they can establish impossibility with a case by case analysis, such as “Well, the first domino has to go either here or there, and if we put it here, then we can only move here, . . .” and

so on. This is quite possible on a  $4 \times 4$  board, and is worth walking through together. You might mention this approach even if nobody suggests it, and give a quick illustration of what it might look like. Regardless, then announce that an  $8 \times 8$  board with opposite corners removed is also impossible, and the thought of trying to analyze all the possible cases is too horrible to consider for long. So it is worthwhile to understand another approach.

d. Have a student with an impossible figure put their example on the board. Then announce that by coloring the squares of the figure black or white in a strategic way, it is possible to prove that no tiling can be found. A student who already knows the solution could present it now; otherwise, brainstorm on some possible colorings. Usually someone suggests the standard checkerboard pattern, but if not just ask for it. (“What about a checkerboard, how is that colored?”) Even if a solution was presented, review it by continuing as follows.

e. Have a student color in the impossible board with a checkerboard pattern, then ask what is special about the way the coloring interacts with a domino. (No matter where a domino is placed, exactly one black and one white square are covered.) (If needed, have a student place a couple of sample dominoes on the board and then ask the class what they notice about the squares that are covered.) So if we could cover the board with seven dominoes, how many white and black squares would appear on the board? (Seven of each.) But how many are there in our impossible board? (Six of one color and eight of the other.) Conclude that this is impossible. Have other students with impossible boards use this approach to explain (to themselves) why no tiling of their figure could be found.

f. A great writing exercise is to have students use coloring to prove that it is impossible to tile a  $6 \times 6$  board in which opposite corners have been removed. If you decide to include this as part of the activity, be sure to write out, in full sentences, an explanation for the impossible board that was presented in class, and have the students copy it down verbatim. (Copying it is a valuable part of the learning process; I would advise against giving them a “pre-fab” proof. This means, of course, that you have to be comfortable with writing the proof in real time! Practice this ahead of time, if necessary.) This will serve as a model for their proof. You might also consider having them turn in a draft, providing feedback, and then getting a polished final paragraph from them a few days later.

g. Finally, I would encourage students to think about where they could take this activity from here, now that we more or less understand the problem of tiling grids with dominoes. Let them brainstorm for a while. Some of the possible avenues of exploration would be to consider different tiling shapes, such as a long domino, or L-shaped piece. Different figures to tile are also possible. One could also consider 3-D tiling or even tiling on other surfaces, such as a cylinder. Some problems along these lines are outlined on the next page.



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### Further Problems.

1. Find a way to tile a  $5 \times 5$  board minus the middle square with  $1 \times 3$  “long dominoes.”
2. Now attempt to tile a  $5 \times 5$  board missing a single corner square. It turns out that this is not possible, but see how close you can come. (How many long dominoes can you fit on before getting stuck?)
3. To show that the previous board cannot be tiled, we will use proof by crayon, naturally. How many colors do you think should be used? And what pattern might be a strategic way of coloring the squares? (You need only come up with a plausible guess here; you don’t have to find a proof using your coloring.)
4. (Harder) Find an explanation, using coloring, that a  $5 \times 5$  board missing a corner square cannot be tiled using long dominoes.
5. There are five distinct shapes which can be constructed using four unit squares on a sheet of graph paper. These are called the *tetrominoes*. To begin, draw all five tetrominoes. Then decide which ones can be used to tile a  $4 \times 4$  board.
6. Show that it is impossible to cover a  $4 \times 5$  rectangle with a complete set of tetrominoes, i.e. using each tetromino once. (HINT: coloring!)
7. Draw a rectangle 7 squares wide and 3 squares high, missing the middle square of the top row. Then find a way to tile this figure with one complete set of tetrominoes. Can you find another interesting shape, consisting of twenty unit squares, which can be built from one complete set of tetrominoes?
8. (Challenge) For each of the five tetrominoes, determine whether or not it can be used to tile a  $6 \times 6$  board. In each case either find a way to perform the tiling or demonstrate that it cannot be done using an appropriate coloring scheme or other argument.
9. (Optional) One can create a  $6 \times 6$  “cylindrical board” by taping two opposite sides of a normal  $6 \times 6$  board together. This effectively eliminates two edges of the board. Which of the tetrominoes can be used to tile the resulting figure?