Oakland/East Bay Teacher's Circle – Parking Functions

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Imagine a one-way cal-de-sac with n parking spots. We'll give the first parking spot the number 1, the next one number 2, etc., down to the last one, number n. Initially they're all free, but there are n cars approaching the street, and they'd all like to park there. To make life interesting, every car has a parking preference, and we record the preferences in a sequence; e.g., if $n = 3$, the sequence $(2, 1, 1)$ means that the first car would like to park at spot number 2, the second car prefers parking spot number 1, and the last car would also like to part at number 1. The street is very narrow, so there is no way to back up. Now each car enters the street and approaches its preferred parking spot; if it is free, it parks there, and if not, it moves down the street to the first available spot. We call a sequence a *parking function* (of length n) if all cars end up finding a parking spot. E.g., the sequence $(2, 1, 1)$ is a parking sequence (of length 3), whereas the sequence $(2, 3, 2)$ is not.

(1) List all parking functions of length 1, 2, and 3.

(2) Come up with some families of parking functions of arbitrary length. (E.g., the sequences $(1, 1, \ldots, 1)$ form one such family.)

(3) Given a sequence $(p_1, p_2, \ldots, p_n)$, let $(q_1, q_2, \ldots, q_n)$ be a permutation of $(p_1, p_2, \ldots, p_n)$ such that $q_1 \leq q_2 \leq \cdots \leq q_n$, i.e., the q_j's are just the p_k's but listed in order. Find a (necessary and sufficient) condition on the q_j's that $(p_1, p_2, \ldots, p_n)$ is a parking function.

(4) Conclude that *any* permutation of a parking function is another parking function.

Now for some variety, suppose instead that we have a one-way roundabout with $n+1$ parking spots. Again n cars would like to park, and again they each have a parking preference, but now they have the luxury of not running into a brick wall at the end of the street—they can just continue on the roundabout. But this means that any sequence $(p_1, p_2, \ldots, p_n)$ of numbers chosen from 1 to $n+1$ will form a *circular parking function* (right?) Also, note that every sequence leaves an empty parking spot.

(5) How many circular parking functions of length n are there?

(6) Prove that the number of circular parking functions that leave parking spot 1 empty equals the number of circular parking functions that leave parking spot 2 empty, it also equals the number of circular parking functions that leave parking spot 3 empty, etc.

(7) Show that a circular parking function that leaves parking spot $n+1$ empty is a parking function.

(8) Conclude a formula for the number of parking functions of length n.