Euler Characteristic Exploration

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Overview. The purpose of this activity is to introduce students to elementary concepts in graph theory in a hands-on, accessible manner. Students will create their own graphs, count certain quantities related to their graphs, make conjectures regarding these quantities, and learn how to explain why their conjecture is true. In the process they will discover the Euler characteristic, a powerful tool for understanding planar graphs.

Materials. You will need one copy of the "Connect the Dots" worksheet for each student. Scotch tape will be useful for displaying the graphs created.

Presentation. The entire exploration will probably last 20 to 30 minutes.

a. If desired, give a little background on the activity. For example, "Today we will be investigating a neat idea in graph theory, a branch of mathematics that wasn't systematically studied until the 1930's. But these graphs are not like the ones made by calculators; they are made by connecting dots with segments. You would never guess it, but there is some great mathematics waiting to be discovered within diagrams like these." Or just jump in with the activity and explain ideas as they arise.

b. Give an example of a connected planar graph on the board first before handing out the worksheet. I like to get the kids involved as much as possible. Ask for some number of vertices from five to ten, plot that many points, explain that you're going to connect some of the dots with segments (but the segments cannot intersect), ask them what they think a "connected diagram" means, and so on. Try to arrange for the segments to enclose at least one region. Then count the number of vertices, faces, and regions in your graph. Make sure that students understand that the segments cannot cross, that all points must be joined into the network, and that the "outside" of the graph counts as a region.

c. Pass out worksheets and have each student make their own connected planar graph. They should then count the number of vertices, edges, and faces in their graph. (Decorating their artwork is fun, but too much decoration will obscure the diagram.) Once they are finished, tape graphs to the wall or display them on a large desk. Once there are enough on display, have students examine the graphs and look for patterns among the numbers.

d. At some point return kids to their seats, restore order, and brainstorm as a class. Ask kids what they noticed. The goal is to discover that $V - E + F = 2$ for all connected planar graphs. Leading questions such as, "Were there usually more vertices, edges, or faces?" and "Were there more edges than the faces and edges combined?" will help, if necessary. If at all possible, let the kids make the final observation themselves.

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e. If time and creative energy permit, begin to discuss why $V - E + F = 2$ holds for any connected planar graph. Emphasize that one of the best ways to understand complicated phenomena is to begin with simple cases and build up from there. Try out the first three problems below. Your class has hit a homerun if any student suggests checking to see whether or not incorporating another edge into an existing graph changes the Euler characteristic.

Further Problems. These problems are well-suited for working in groups of two or three. There are probably too many here for a single student to complete as an assignment.

1. The simplest possible graph consists of a single vertex in the plane. Compute V (the number of vertices, or points), E (the number of edges, or segments), and F (the number of faces, or regions) in this case. What does $V - E + F$ equal?

2. Draw a connected graph which has only one edge. Then draw a connected graph with two edges. Calculate V, E, and F for your graphs. What is the Euler characteristic for each?

3. There are three different connected graphs having exactly three edges. Draw examples of these, and compute the Euler characteristic in each case.

4. Suppose we take a graph and add a single extra vertex off to the side, then join it to some existing vertex with an edge. How does this process affect the total number of vertices, edges, and faces? Does the Euler characteristic change?

5. Now consider what happens if we take two vertices which are already part of a given graph (but which are not yet connected) and draw a new edge between them. How does this affect the total number of vertices, edges, and faces? Does the Euler characteristic change?

6. Find a way to draw four points in the plane so that when you connect every pair of points with a line segment, none of the segments intersect.

7. Experiment to see what happens to the Euler characteristic if the graph is not connected. First draw three graphs, each with a different number of "connected pieces." Compute $V - E + F$ in each case. What aspect of your graph does the Euler characteristic measure?

8. In a certain small country there are villages, expressways, and fields. Expressways only lead from one village to another and do not cross one another, and it is possible to travel from any village to any other village along the expressways. Each field is completely enclosed by expressways and villages. If there are ten villages and sixteen expressways, then how many fields are there in this country? Why is the answer the same no matter how the map looks?