The Game of Criss-Cross

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Overview. The purpose for having students play the game of Criss-Cross is to motivate them to explain the underlying mathematical reason governing who wins or loses. This exploration should lead the students to form, test, and ultimately prove conjectures about how to win at Criss-Cross. The game illustrates a beautiful application of the Euler characteristic and gives them practice at elementary counting techniques as well.

Materials. You will need one copy of "The Game of Criss-Cross" worksheet for each student. Scotch tape will be useful for displaying the finished games. Players can use colored pens for marking their segments on the game board.

Presentation. The entire exploration will probably last 25 to 30 minutes.

a. Illustrate a sample game on the board first, using only two interior points to keep it short. Note that the first player wins your game. Encourage students to space their points out on their game boards. Also point out that segments cannot pass through a third point in the diagram. (That would be like drawing two segments that line up.) Make sure students realize that drawing the sides of the outer triangle are legal moves.

b. Have each student write their name on their game board, then plot up to seven other "fat" points inside the triangle formed by the three given vertices. Remind them not to draw in the edges of the triangle yet. Each student must then find an opponent who will play against them on their game board; students always move first on their own game board. After all the games have been played (which will require at least two rounds), have each student count up the vertices, edges, and faces on their game board. Display all the results.

c. Give students a few minutes to examine all of the game boards and look for patterns. They should try to figure out how to predict who will win, but they should also be on the lookout for any other interesting patterns among the numbers. During this time nobody is allowed to shout out observations! Then reconvene and see what the class has come up with. The goal is for the kids to conjecture that the first player will win if there are an odd number of total points on the board, while the second player will win if there are an even number of points. They can then retrieve their game boards. Guide them to discover that $3F = 2E$ on each game board as well, since they may not notice this at first.

d. Explain that there are two important ingredients to proving their conjecture about who wins. Ask them what the gameboard reminds them of. (The Euler characteristic, of course.) The second is the fact that $3F = 2E$. Have them work out why this is the case by asking what they notice about each of the faces. (They are all triangles.) In fact, even the "outer region" is bounded by three sides. Ask why this occurs. (Otherwise another play could have been made in the game.) Suggest that because of this fact a game board with four regions would have twelve edges all together. Hopefully someone will object and point out that edges get counted more than once. Ask which edges, and eventually conclude that every edge is counted exactly twice, since each edge is a side of precisely two faces. Hence $3F = 2E$. (If necessary, draw a single point inside a triangle, connect all pairs of points, and note that here four regions are defined by only six edges. What went wrong? Then continue as above.)

e. Analyze a game board with four interior points as follows. Start with $V - E + F$ and have students point out that this quantity equals 2 (Euler characteristic), that $V = 7$ (just count), and that $F = \frac{2}{3}E$ (this may take some leading). Then solve the resulting equation to obtain $E = 15$. Emphasize that this is surprising—no matter where the points are placed or how the segments are played, there must be exactly 15 segments in the end. Double-check with actual game boards in the room, if anyone chose four interior points for their game. Ask who wins a game with 15 moves. (The first player.) Make any desired concluding remarks.

Further Problems.

1. Let's consider the situation in which five additional points are drawn inside the triangle of a Criss-Cross game board. Imagine that a game is played on this board, and that there are V vertices, E edges, and F faces at the conclusion of the game. What is the value of V? What is the value of $V - E + F$?

2. On the completed game board of the previous problem, how do we know that $3F = 2E$? (This was covered during our in-class exploration; just briefly summarize what was said then.)

3. Now figure out values for E and F so that $3F = 2E$ and $8 - E + F = 2$. You can either use algebra or educated guessing. Once you know E you know the total number of moves in the game. So, who won the game, the first or second player?

4. Now put all the ideas of the previous problems together to figure out who wins a game of Criss-Cross in which ten additional points are drawn inside the triangle of a standard game board. Justify your answer using the same steps as above.

5. (Harder) In general, prove that $E = 3V - 6$ on a completed game board with a total of V vertices. Use this relationship to predict which player wins at Criss-Cross based on the number of vertices at the outset.

6. (Harder) Suppose that we change the outer perimeter of a Criss-Cross board so that it consists of four points at the corners of a square instead of three points in a triangle. Explain why $3F + 1 = 2E$ on a completed game board of this type. Now suppose that we place nine additional points in the interior and play a game. Who will win, and why? (Compare your result to problem 4, in which there were also a total of thirteen points on the game board.)

7. Plot five points on a sheet of paper and draw an edge connecting every pair of points. How many edges are needed? Experiment a bit to decide whether it is possible to position the points so that none of these edges cross.

8. Suppose that someone claims to have a diagram which solves the previous problem. Even though you can't see the picture, what must be their values for V, E, and $V - E + F$? From here deduce the number of faces F in their diagram.

9. (Challenge) Explain why in general $3F \leq 2E$. Use this relationship to prove that the numbers in the previous problem are contradictory, meaning that the hypothetical diagram cannot really exist. (In other words, the last half of problem 7 is impossible.)