(MOSTLY) SIMPLE (MOSTLY AREA) PROBLEMS

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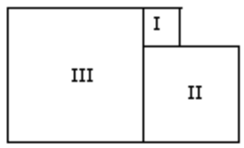
I. EASY PROBLEMS

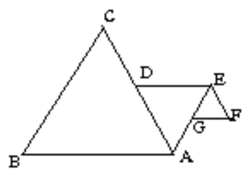
1. (AJHSME 1995)



Figures I, II and III are squares. The perimeter of I is 12 and the perimeter of II is 24. The perimeter of III is

A) 9 B) 18 C) 36 D) 72 E) 81

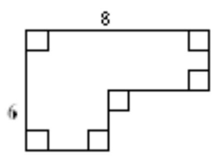
2. (AMC 8 2000)



Triangles ABC, ADE, and EFG are all equilateral. Points D and G are midpoints of AC and AE, respectively. If AB = 4, what is the perimeter of figure ABCDEFG?

A) 12 B) 13 C) 15 D) 18 E) 21

3. (AJHSME, 1986)



The perimeter of the polygon shown is

A) 14 B) 20 C) 28 D) 48

E) cannot be determined from the information given

4. (AMC 8 2000)

In order for Mateen to walk a kilometer (1000 m) in his rectangular backyard, he must walk the length 25 times or walk its perimeter 10 times. What is the area of Mateen's backyard in square meters?

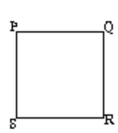
A) 40 B) 200 C) 400 D) 500 E) 1000

5. (AMC 8 1999)

A rectangular garden 50 feet long and 10 feet wide is enclosed by a fence. To make the garden larger, while using the same fence, its shape is changed to a square. By how many square feet does this enlarge the garden?

A) 100 B) 200 C) 300 D) 400 E) 500

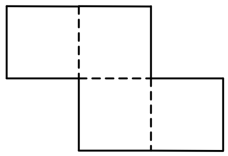
6. (AJHSME 1998)



Let PQRS be a square piece of paper. P is folded onto R and then Q is folded onto S. The area of the resulting figure is 9 square inches. Find the perimeter of square PQRS.

A) 9 B) 16 C) 18 D) 24 E) 36

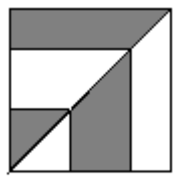
7. (AJHSME 1990)



The area of this figure is 100 cm^2. (The figure consists of four identical squares.) Its perimeter is

A) 20 cm B) 25 cm C) 30 cm D) 40 cm E) 50 cm

8. (AJHSME 1990)

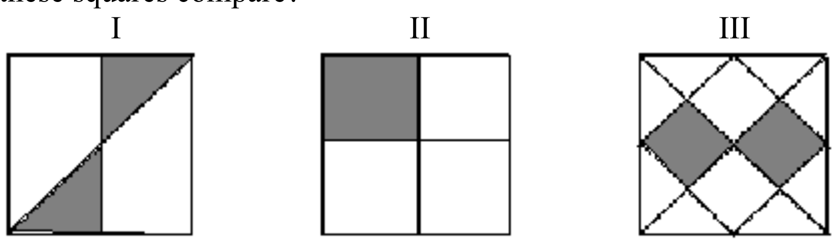


What fraction of the square is shaded?

A) 1/3 B) 2/5 C) 5/12 D) 3/7 E) ½

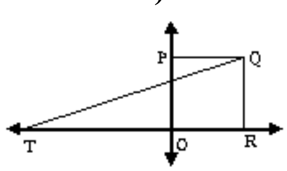
9. (AJHSME 1994)

Each of the three large squares shown below is the same size. Segments that intersect the sides of the squares intersect at the midpoints of the sides. How do the shaded areas of these squares compare?



A) The shaded areas in all three are equal.
B) Only the shaded areas of I and II are equal.
C) Only the shaded areas of I and III are equal.
D) Only the shaded areas of II and III are equal.
E) The shaded areas of I, II and III are all different.

10. (AJHSME 1996)

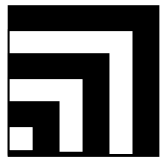


(Figure is drawn NOT TO SCALE.)

Figure OPQR is a square. Point O is the origin, and point Q has coordinates (2, 2). What are the coordinates for T so that the area of triangle RQT equals the area of square OPQR?

A) $(-6,0)$ B) $(-4,0)$ C) $(-2,0)$ D) $(2,0)$ E) $(4,0)$

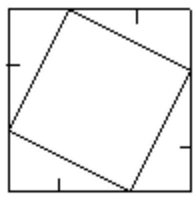
11. (AJHSME 1997)



What fraction of this square region is shaded? Stripes are equal in width, and the figure is drawn to scale.

A) 5/12 B) ½ C) 7/12 D) 2/3 E) 5/6

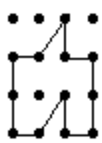
12. (AJHSME 1997)



Each side of the large square in the figure is trisected (divided into three equal parts). The corners of an inscribed square are at these trisection points, as shown. The ratio of the area of the inscribed square to the area of the large square is

A) $\sqrt{3}/3$ B) 5/9 C) 2/3 D) $\sqrt{5}/3$ E) 7/9

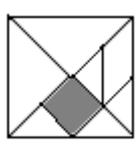
13. (AJHSME 1998)



Dots are spaced one unit apart, horizontally and vertically. The number of square units enclosed by the polygon is

A) 5 B) 6 C) 7 D) 8 E) 9

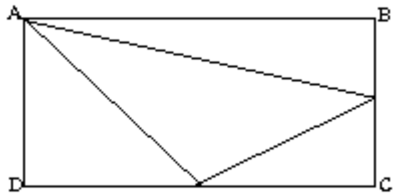
14. (AJHSME 1998)



What is the ratio of the area of the shaded square to the area of the large square? (The figure is drawn to scale.)

A) 1/6 B) 1/7 C) 1/8 D) 1/12 E) 1/16

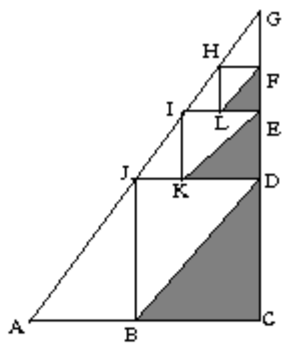
15. (AMC 8 2000)



The area of rectangle ABCD is 72. If point A and the midpoints of BC and CD are joined to form a triangle, the area of that triangle is

A) 21 B) 27 C) 30 D) 36 E) 40

16. (AMC 8 1999)



Points B, D, and J are midpoints of the sides of right triangle ACG. Points K, E, I are midpoints of the sides of triangle JDG, etc. If the dividing and shading process is done 100 times (the first three are shown) and AC = CG = 6, then the total area of the shaded triangles is nearest

A) 6 B) 7 C) 8 D) 9 E) 10

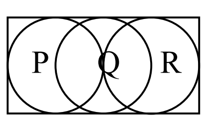
17. (AJHSME 1997)

The area of the smallest square that will contain a circle of radius 4 is

A) 8 B) 16 C) 32 D) 64 E) 128

18. (AJHSME 1995)

A B

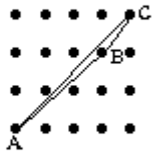


D C

Three congruent circles with centers P, Q and R are tangent to the sides of rectangle ABCD as shown. The circle centered at Q has diameter 4 and passes through points P and R. The area of the rectangle is

A) 16 B) 24 C) 32 D) 64 E) 128

19. (AJHSME 1996)



The horizontal and vertical distances between adjacent points equal 1 unit. The area of triangle ABC is

A) ¼ B) ½ C) ¾ D) 1 E) 5/4

20. (AMC 8 2000)

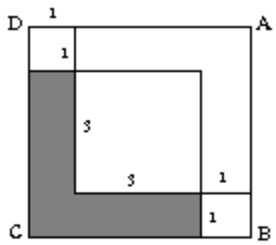
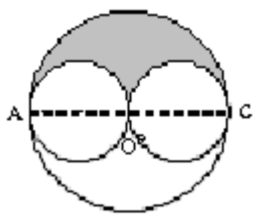


Figure ABCD is a square. Inside this square three smaller squares are drawn with side lengths as labeled. The area of the shaded L-shaped region is

A) 7 B) 10 C) 12.5 D) 14 E) 15

21. (AJHSME, 1986)



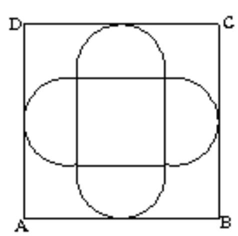
The larger circle has diameter AC. The two small circles have their centers on AC and just touch at O, the center of the large circle. If each small circle has radius 1, what is the value of the ratio of the area of the shaded region to the area of one of the small circles?

A) between ½ and 1 B) 1 C) between 1 and 3/2 D) between 3/2 and 2
E) cannot be determined from the information given

22. (AJHSME 1994)
The perimeter of one square is 3 times the perimeter of another square. The area of the larger square is how many times the area of the smaller square?

A) 2 B) 3 C) 4 D) 6 E) 9

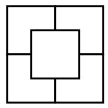
23. (AJHSME 1994)



Around the outside of a 4 by 4 square, construct four semicircles (as shown in the figure) with the four sides of the square as their diameters. Another square, ABCD, has its sides parallel to the corresponding sides of the original square, and each side of ABCD is tangent to one of the semicircles. The area of the square ABCD is

A) 16 B) 32 C) 36 D) 48 E) 64

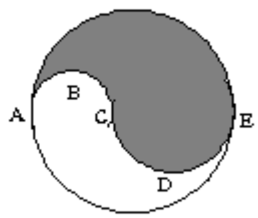
24. (AJHSME 1995)



The area of each of the four congruent L-shaped regions of this 100-inch by 100-inch square is 3/16 of the total area. How many inches long is the side of the center square?

A) 25 B) 44 C) 50 D) 62 E) 75

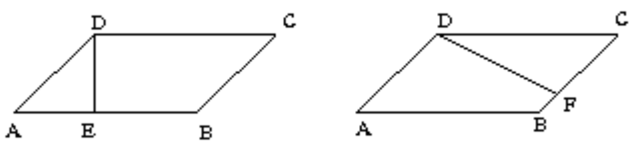
25. (AJHSME 1997)



Diameter ACE is divided at C in the ratio 2:3. The two semicircles, ABC and CDE, divide the circular region into an upper (shaded) region and a lower region. The ratio of the area of the upper region to that of the lower region is

A) 2:3 B) 1:1 C) 3:2 D) 9:4 E) 5:2

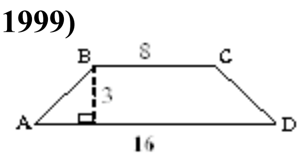
26. (AJHSME 1995)



In parallelogram ABCD, DE is the altitude to the base AB and DF is the altitude to the base BC. [**Note**: *Both pictures represent the same parallelogram*.] If DC =12, EB = 4 and DE = 6, then DF =

A) 6.4 B) 7 C) 7.2 D) 8 E) 10

27. (AMC 8 1999)

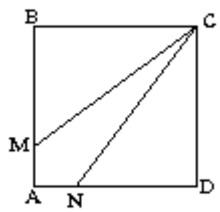


In trapezoid ABCD, the sides AB and CD are equal. The perimeter of ABCD is

A) 27 B) 30 C) 32 D) 34 E) 48

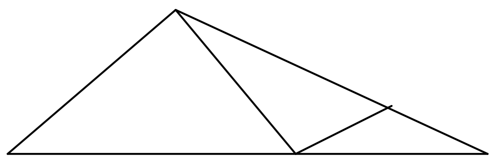
28. (AMC 8 1999)

Square ABCD has sides of length 3. Segments CM and CN divide the square's area into three equal parts. How long is segment CM?



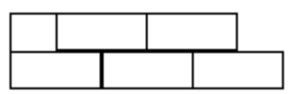
A) $\sqrt{10}$ B) $\sqrt{12}$ C) $\sqrt{13}$ D) $\sqrt{14}$ E) $\sqrt{15}$

29. (AJHSME 1998)



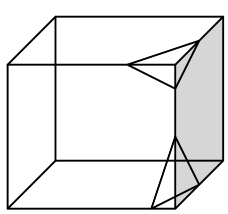
How many triangles are in this figure? (Some triangles may overlap other triangles.)
A) 9 B) 8 C) 7 D) 6 E) 5

30. (AMC 8 2000)
A block wall 100 feet long and 7 feet high will be constructed using blocks that are 1 foot high and either 2 feet long or 1 foot long (no blocks may be cut). The vertical joins in the blocks must be staggered as shown, and the wall must be even on the ends. What is the smallest number of blocks needed to build this wall?



A) 344 B) 347 C) 350 D) 353 E) 356

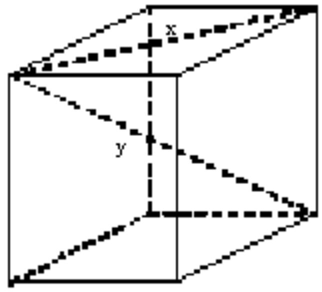
31. (AJHSME 1990)



Each corner of a rectangular prism is cut off. Two (of the eight) cuts are shown. How many edges does the new figure have?
A) 24 B) 30 C) 36 D) 42 E) 48

32. (AJHSME 1998)
A 4 x 4 x 4 cubical box contains 64 identical small cubes that exactly fill the box. How many of these small cubes touch a side or the bottom of the box?
A) 48 B) 52 C) 60 D) 64 E) 80

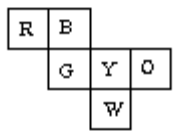
33. (AJHSME 1997)



A cube has eight vertices (corners) and twelve edges. A segment, such as x, which joins two vertices not joined by an edge is called a diagonal. Segment y is also a diagonal. How many diagonals does a cube have?

A) 6 B) 8 C) 12 D) 14 E) 16

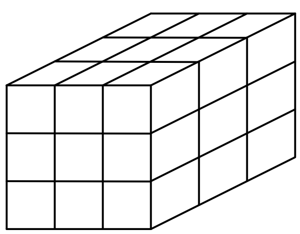
34. (AMC 8 1999)



Six squares are colored, front and back, (R=red, B=blue, O=orange, Y=yellow, G=green, and W=white). They are hinged together as shown, then folded to form a cube. The face opposite the white face is

A) B B) G C) O D) R E) Y

35. (AJHSME 1997)



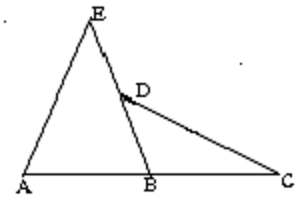
Each corner cube is removed from this 3 cm x 3 cm x 3 cm cube. The surface area of the remaining figure is

A) 19 sq. cm B) 24 sq. cm C) 30 sq. cm D) 54 sq. cm E) 72 sq. cm

36. (AMC 8 2000)
A cube has edge length 2. Suppose that we glue a cube of edge length 1 on top of the big cube so that one of its faces rests entirely on the top face of the larger cube. The percent increase in the surface area (sides, top, and bottom) from the original cube to the new solid formed is closest to:

A) 10 B) 15 C) 17 D) 21 E) 25

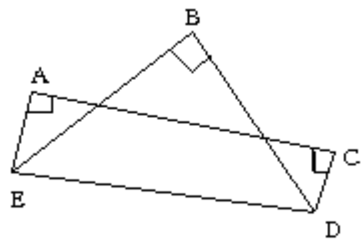
37. (AJHSME 1994)



If $\angle A = 60°$, $\angle E = 40°$ and $\angle C = 30°$, then $\angle BDC =$

A) 40° B) 50° C) 60° D) 70° E) 80°

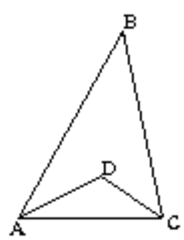
38. (AJHSME 1995)



In the figure, $\angle A$, $\angle B$, and $\angle C$ are right angles. If $\angle AEB = 40°$ and $\angle BED = \angle BDE$, then $\angle CDE =$

A) 75° B) 80° C) 85° D) 90° E) 95°

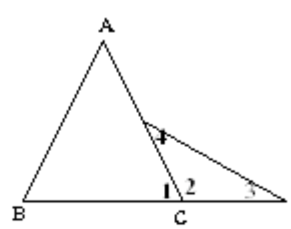
39. (AJHSME 1996)



The measure of angle ABC is 50°, AD bisects angle BAC, and DC bisects angle BCA. The measure of angle ADC is

A) 90° B) 100° C) 115° D) 122.5° E) 125°

40. (AJHSME 1997)



$\angle ABC = 70°$, $\angle BAC = 40°$, $\angle 1 + \angle 2 = 180°$, $\angle 3 = \angle 4$. Find $\angle 4$

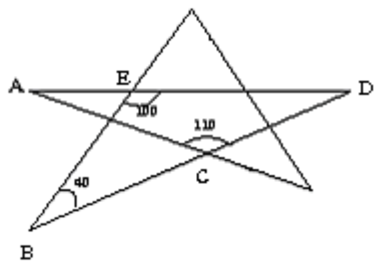
A) 20° B) 25° C) 30° D) 35° E) 40°

41. (AMC 8 1999)

What is the degree measure of the smaller angle formed by the hands of a clock at 10 o'clock?

A) 30 B) 45 C) 60 D) 75 E) 90

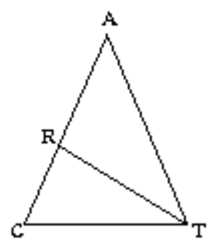
42. (AMC 8 1999)



$\angle B = 40°$, $\angle BED = 100°$, $\angle ACD = 110°$. The degree measure of angle A is

A) 20 B) 30 C) 35 D) 40 E) 45

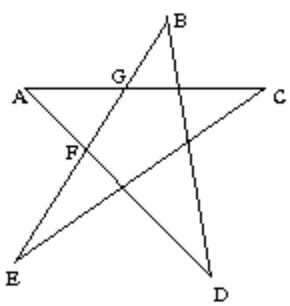
43. (AMC 8 2000)



In triangle CAT, we have $\angle ACT = \angle ATC$ and $\angle CAT = 36°$. If TR bisects $\angle ATC$, then $\angle CRT =$

A) $36°$ B) $54°$ C) $72°$ D) $90°$ E) $108°$

44. (AMC 8 2000)



If $\angle A = 20°$ and $\angle AFG = \angle AGF$, then $\angle B + \angle D =$

A) $48°$ B) $60°$ C) $72°$ D) $80°$ E) $90°$

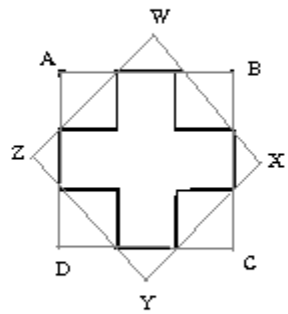
45. (AJHSME 1996)

Points A and B are 10 units apart. Points B and C are 4 units apart. Points C and D are 3 units apart. If A and D are as close as possible, then the number of units between them is

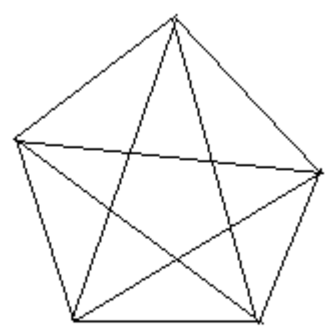
A) 0 B) 3 C) 9 D) 11 E) 17

II. INTERMEDIATE PROBLEMS

I 1. A cross shape with twelve edges, each of length 1, is inscribed in two squares ABCD and WXYZ, as shown. Find the area of each of these two squares.



I 2. The diagram shows a **regular** pentagon, a pentagram (the five-pointed star) and a smaller pentagon inside. The perimeter of the inner pentagon is 5 cm and the perimeter of the pentagram is 10*x* cm. Show that the perimeter of the outer pentagon is $5x^2$ cm.



NOT TO SCALE

I 3. The rectangle ABCD has AB = 15 and AD = 10. P is the point inside the rectangle for which AP =10 and DP = 12. Find the angle DPC.

I 4. Triangle ABC has AB = AC = 5 cm and BC = 6 cm. From any point P, inside or on the boundary of this triangle, line segments are drawn at right angles to the sides; the lengths of these line segments are x, y and z cm.

(a) Find the largest possible value of the total $x + y + z$ and find the positions of P where this largest total occurs.
(b) Find the smallest value of the total $x + y + z$ and find the positions of P where this smallest total occurs.
(c) What if ABC is a general (non-isosceles) triangle?

I 5. In triangle ABC, $\angle ACB$ is a right angle, BC = 12 and D is a point on AC such that AD = 7 and DC = 9. The perpendicular from D to AB meets AB at P and the perpendicular from C to BD meets BD at Q. Calculate:

(a) The ratio of the area of triangle BCD to the area of triangle BAD.
(b) The ratio of the length of QC to the length of PD.

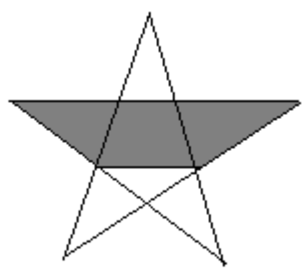
I 6. A square has one corner folded over to create a pentagon. The three shorter sides of the pentagon which is formed are all the same length. Find the area of the pentagon as a fraction of the area of the original square.

I 7. A square is inscribed inside a quadrant of a circle of radius 10 cm. Calculate the area of the square.

I 8. In a triangle the length of one side is 3.8 cm and the length of another side is 0.6 cm. Find the length of the third side if it is known that it is an integer (when expressed in centimeters).

I 9. Using a pen and a straight edge, draw on a square grid a square whose area is
(a) twice the area of one square of the grid;
(b) 5 times the area of one square of the grid.

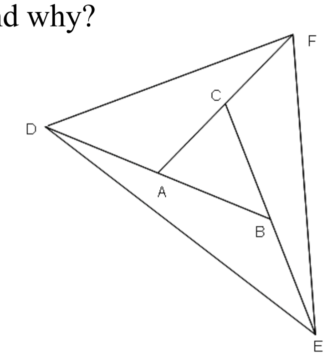
I 10. Find the area of the shaded region as a fraction of the area of the entire regular pentagram.



NOT TO SCALE

I 11. Is it true that a half liter bottle of Coke is proportional to a liter bottle of Coke, i.e., one can be obtained from another by multiplying all lengths by the same factor?

I 12. Suppose that triangle ABC has an area of 1. Plot points D, E, and F so that A is the midpoint of BD, B is the midpoint of CE, and C is the midpoint of AF. What is the area of triangle DEF and why?



III. HARDER PROBLEMS

For some problems in this section, we will need to know the meaning of the term 'convex'. A *convex* planar figure is the intersection of a number (finite or infinite) of half-planes. The intersection of a finite number of half-planes is a *convex polygon*. (Equivalently, a figure is convex if for any two points A and B of the figure, the entire line segment AB belongs to the figure.)

H1. Suppose that $ABCD$ is a convex quadrilateral. Let's extend its sides AB, BC, CD, and DA so that B is the midpoint of AB_1, C is the midpoint of BC_1, D is the midpoint of CD_1, and A is the midpoint of DA_1. If the area of the quadrilateral $ABCD$ is 1, find the area of $A_1B_1C_1D_1$.

H2. Let $ABCD$ be a trapezoid with $AD \parallel BC$ and let O be the point of intersection of the diagonals AC and BD. Prove that the triangles AOB and DOC have equal areas.

H3. Let ABC be a triangle and suppose that P is an arbitrary point on AB. Find a line through P that divides ABC into two regions of equal areas.

H4. Let $ABCD$ be a convex quadrilateral. Find a line through the vertex A that divides $ABCD$ into two regions of equal areas.

H5. Two lines trisect each of two opposite sides of a convex quadrilateral. Prove that the area of the part of the quadrilateral contained between the lines is one third of the area of the quadrilateral.

H6. Suppose $ABCD$ is a convex quadrilateral. Two points are given on each of the four sides of this quadrilateral: K and M on AB, P and R on BC, N and L on CD, S and Q on DA. Each pair of these points trisects the corresponding side, so that $AK = KM = MB$; $BP = PR = RC$; $CN = NL = LD$; $DS = SQ = QA$. Let A_1 be the point of intersection of KL and PQ; B_1 be the point of intersection of MN and PQ; C_1 be the point of intersection of MN and RS; and let D_1 be the point of intersection of KL and RS. Prove that the area of $A_1B_1C_1D_1$ is 1/9 of that of $ABCD$.

H 7. In a certain country, there are 100 airports and all the distances between them are different. An airplane takes off from each airport and lands at the closest airport. Prove that none of the 100 airports receives more than 5 planes.

H 8. n points are placed in a plane in such a way that the area of every triangle with vertices at any three of these points is at most 1. Prove that all these points can be covered by a triangle with the area of 4.

H 9. Is it possible to place 1000 line segments in a plane in such a way that every endpoint of each of these line segments is at the same time an inner point of another segment?

H 10. Suppose that points A, B, C, and D are coplanar but non-collinear. Prove that at least one of the triangles formed by these points is not acute.

H 11. Prove that

(a) any convex polygon of area 1 can be covered by a parallelogram of area 2;

(b) a triangle of area 1 cannot be covered by a parallelogram of area less than 2.

H 12. (a) Suppose that there are four convex figures in a plane, and every three of them have a common point. Prove that all four figures have a common point.

(b) Suppose that there are n convex figures in a plane, and every three of them have a common point. Is it necessarily true that all n figures have a common point?

H 13. A number of lines segments lie in a plane in such a way that for any three of them there exists a line intersecting them. Prove that there exists a line intersecting all these segments.

H 14. Is it true that for every pentagon it is possible to find at least two sides such that the pentagon belongs to the intersection of exactly two half-planes determined by these sides?

H 15. (a) Draw a polygon and a point P inside this polygon so that none of the sides is completely visible from the point P.

(b) Draw a polygon and a point P outside of this polygon so that none of the sides is completely visible from the point P.

H 16. (a) Prove that every *n*-gon (with $n \geq 4$) has at least one diagonal that is completely contained inside the *n*-gon.

(b) Find the least possible number of such diagonals.

(c) Prove that every polygon can be cut into triangles by non-intersecting diagonals.

(d) Suppose that a polygon is cut into triangles by non-intersecting diagonals. Prove that it is possible to color the vertices of the polygon using three colors in such a way that all three vertices of each of the triangles are different.

Answers, Hints, Solutions

I. Easy Problems

(1) Answer Key

1.C; 2.C; 3.C; 4.C; 5.D; 6.D; 7.E; 8.E; 9.A; 10.C; 11.C; 12.B; 13.B; 14.C; 15.B; 16.A; 17.D; 18.C; 19.B; 20.A; 21.B; 22.E; 23.E; 24.C; 25.C; 26.C; 27.D; 28.C; 29.E; 30.D; 31.C; 32.B; 33.E; 34.A; 35.D; 36.C; 37.B; 38.E; 39.C; 40.D; 41.C; 42.B; 43.C; 44.D; 45.B

(2) General Remarks

All problems in this section are taken from either AMC 8 or its predecessor, AJHSME. All of these problems are multiple choice and each one is expected to be solvable in less than two minutes – which fully justifies the title of the section.

Problems 1-3 are about perimeter; 4-7 deal with perimeter and area, and only area of a rectangle is used. In problems 8-16 no formula for area – even that of a rectangle is hardly needed – just the idea that congruent figures have equal area will work. In problems 17-20 formulas for the area of a rectangle and a triangle are needed. Problems 21-25 have amazingly nice solutions which do not require knowing any formulas – instead, the notion of scaling can be used. In fact, these problems can be used as an incentive to explore this notion. In problems 26-28 Pythagorean Theorem is useful, yet only in 28 it is used in general while in 26 and 27 only a 3-4-5 triangle (or its scaled version) is needed.
Problems 29-34 are actually counting or combinatorics problems more than anything else even though they ostensibly use geometric setting, even going into 3-d; while problems 35 and 36 go back to area – this time applied to 3-d objects. Problems 37-44 are all about angles and require a very minimal acquaintance with properties of angles of a triangle. Problem 45 is added as an illustration of how the choices provided make it so much easier to solve some problems.

(3) Some more detailed hints or solutions.

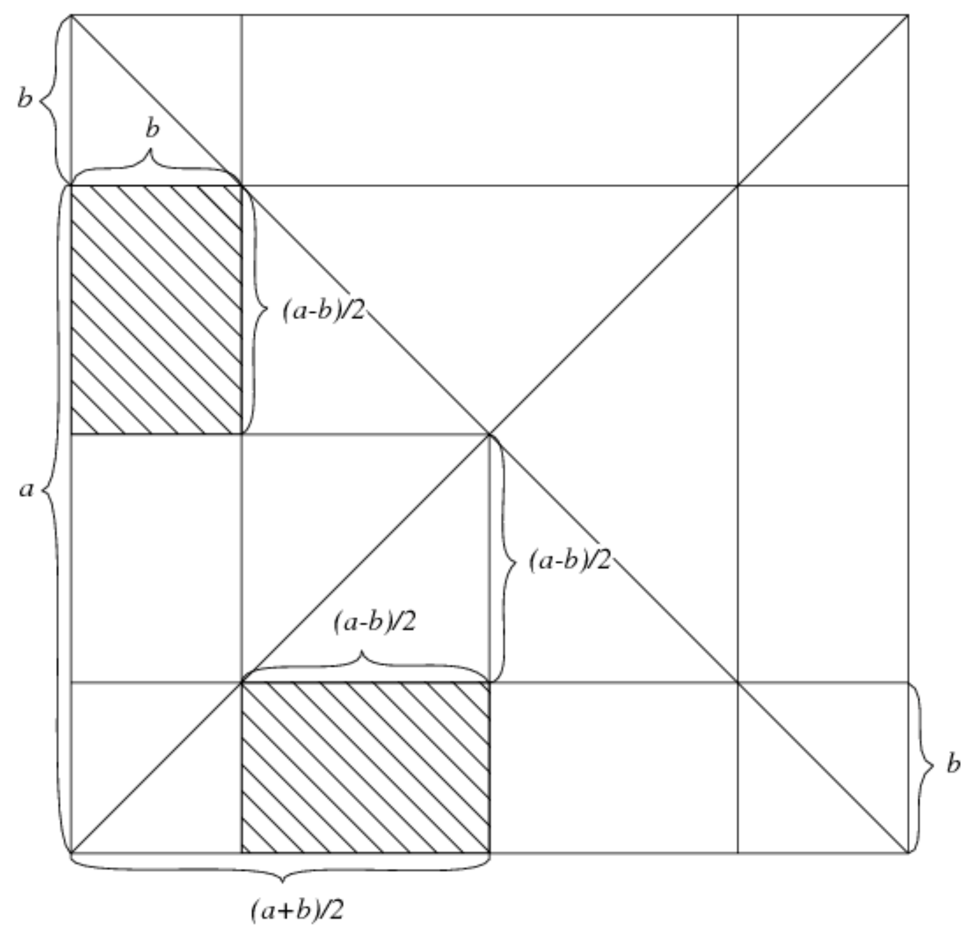
1. Suppose that sides of squares I and II are painted R, B, W, Y. We can arrange 4 copies of each of these 2 squares around square III in such a way that square III also has R, B, W, and Y sides. Thus its perimeter is the sum of perimeters of I and II, i.e., 36.

2. Figure ABCDEFG has 2 sides of length 4, 2 sides of length 2, and 3 sides of length 1, so its perimeter is $2 \cdot 4 + 2 \cdot 2 + 3 \cdot 1 = 15$.

3. Imagine that the figure is made of metal rods hinged at the corners. Take the inside corner and rotate it outward. Eventually we will get an 8 by 6 rectangle. The perimeter of the original figure is exactly the same as the perimeter of this rectangle, i.e., $2 \cdot 8 + 2 \cdot 6 = 28$.
Note: the idea of transforming a given figure in such a way that the required quantity is an invariant which it is easier to find for the new transformed figure is a very fruitful one. We will make a frequent use of this method.

4. $25L = 1000$, so $L = 40$. $10 \cdot 2 \cdot (40 + W) = 1000$, so $W = 10$. Thus the area is $40 \cdot 10 = 400$.

5. This problem can be easily generalized. If the sides of the given rectangle are *a* and *b*, then the area of the square of equal perimeter is exactly $\left(\frac{a-b}{2}\right)^2$ larger than the area of the rectangle. This is easy to see algebraically:
$\left(\frac{2(a+b)}{4}\right)^2 - ab = \frac{a^2 + 2ab + b^2 - 4ab}{4} = \frac{a^2 - 2ab + b^2}{4} = \left(\frac{a-b}{2}\right)^2$. Alternately, this can be seen geometrically starting with a square of side $a + b$, then drawing its diagonals, etc. – see the diagram below.



6. Area of PQRS is $4 \cdot 9 = 36$, hence its side is 6, and its perimeter is $4 \cdot 6 = 24$.

7. The area of each small square is 100/4 = 25, so its side is 5. The number of sides is 10 (they can be simply counted, or we can calculate their number as $4 \cdot 4 - 3 \cdot 2 = 10$ since 4 squares have $4 \cdot 4$ sides, but 3 of these sides each counted twice are not included in the perimeter).

8. Each shaded region has a congruent unshaded counterpart and since congruent regions have equal area so the total area of shaded regions is exactly ½ of the total area.

9. If we draw a horizontal midline in figure I it becomes clear that the two shaded triangles can be rearranged to coincide with the shaded square of figure II. Likewise, if we draw both vertical and horizontal midlines in figure III, we can see that 4 shaded triangles thus obtained can be rearranged to coincide with the shaded square of figure II. Thus all shaded regions have equal areas.

10. If the point T is such that the triangles formed by the line TQ inside and outside of the square OPQR are congruent, then obviously the condition of the problem is satisfied. In order for this to happen, the length of TO must be the same as the length of PQ, so point T must have coordinates $(-2, 0)$.

11. Placing the figure on a grid, we see that there are 21 shaded squares while the entire figure consists of 36 squares. thus the answer is $\frac{21}{36} = \frac{7}{12}$.

12. If we draw 2 vertical and 2 horizontal lines thus dividing the figure into 9 equal squares, we can see that the 4 triangles can be rearranged into two 1-by-2 rectangles which cover 4 of these squares. Thus the ratio is $(9-4)/9 = 5/9$.

13. By moving the top triangle down we see that the area is $3 \cdot 2 = 6$.

14. Dividing the shaded square by its vertical diagonal we see that it consists of two triangles, while a fourth of the square consists of 4 of these triangles. Hence the shaded square is 1/8 of the entire figure.

15. Area of the triangle ACG is $\frac{6 \cdot 6}{2} = 18$. Triangles ABJ, BJD, and DCD are congruent, so the area of BCD is 1/3 of that of ACDJ. Similarly, the area of KDE is 1/3 of the area of JDEI; the area of LEF is 1/3 of the area of IEFH, etc. Hence the total area of the shaded triangles is a bit less than 18, i.e., it's close to and less than 6.

16. The side of such a square is 8 so its area is 64.

17. AB = 8, BC = 4, so the area is 32.

18. The area is half the area of the entire rectangle minus the area of the triangle below AB minus the area of the rectangle and triangle below BC, i.e.,
$\frac{4 \cdot 3}{2} - \frac{2 \cdot 3}{2} - 1 \cdot 2 - \frac{1 \cdot 1}{2} = \frac{1}{2}$. (Note: alternately, by Pick's Theorem the area is $0 + \frac{3}{2} - 1 = \frac{1}{2}$.)

19. Choose any one of the following self-evident computations:

$$5 \cdot 5 - 2(1 \cdot 1) - 4 \cdot 4 = 7 \text{, or}$$
$$2(3 \cdot 1) + 1 \cdot 1 = 7 \text{, or}$$
$$\frac{5 \cdot 5 - 3 \cdot 3 - 2(1 \cdot 1)}{2} = 7$$

20. Drawing the horizontal middle line we see that the top outer triangle is ½ of the top half of the rectangle ABCD so its area is $\frac{72}{4} = 18$. Likewise, drawing the vertical middle line demonstrates that the area of the bottom left outer triangle is $\frac{72}{4} = 18$. Using both middle lines we see that the area of the bottom right outer triangle is $\frac{72}{8} = 9$. Thus the area of the triangle in question is $72 - 18 - 18 - 9 = 27$.

21. Imaging having in a plane a rectangular grid. If we stretch the lane vertically by a factor of 2, one square will be transformed into two squares. If we now stretch the plane by a factor of 2 horizontally, these two squares will be stretched into four. Thus stretching the plane in both directions by a factor of 2 results in multiplying the area by 4. This kind of reasoning can be generalized in various ways. What happens if the plane is stretched in both horizontal and vertical direction by a factor of *m*? What happens if it's stretched in two mutually perpendicular directions other than vertical and horizontal? What if these two directions form and angle other than 90 degrees? What if the stretching factors differ along two directions? This can be an interesting topic to let students investigate.

Now let's go back to the problem. Suppose that the area of each small circle is 1. Since in order to produce the big circle from a small one the stretching factor in both direction is 2, so the area of the big circle is 4. From symmetry the area of the shaded region is exactly the same as that of the region inside the big circle below two small circles. Hence the area of the shaded region is $(4 - 1 - 1)/2 = 1$. Hence the ratio is 1/1=1.

22. The scaling factor in each direction is 3 hence the area of the big square is $3^2 = 9$ that of the small one. (Alternately, just draw a 3 by 3 square on the grid.)

23. Big square's side is 8, so its area is 64. (Scaling can be used, too.)

24. Area of the small square is $1 - 4\left(\frac{3}{16}\right) = \frac{4}{16} = \frac{1}{4}$ of the area of the big one. Thus the linear stretching factor is ½, so the side of the inner square is $\frac{1}{2}(100) = 50$.

25. Let's assume that the area of a circle whose radius is ½ of the length AC is 1. Then the area of the circle with diameter AC is 4, the area of the circle with diameter CE is 9, and the area of the circle with diameter AE is 25. From symmetry, the area of the part of the shaded region outside of the circle with diameter CE is $\frac{25 - 4 - 9}{2} = 6$. Hence the area

of the shaded region is 6+9=15 while the area of the unshaded region is 4+6=10. Hence the required ratio is 15:10=3:2.

Note: What if the ratio AC:CE = *a*:*b*? Would it be true that the ratio of the area of the upper region to that of the lower region is *b*:*a*? Indeed, similarly to the above computations the ratio will be $\dfrac{\dfrac{(a+b)^2-a^2-b^2}{2}+b^2}{a^2+\dfrac{(a+b)^2-a^2-b^2}{2}}=\dfrac{ab+b^2}{a^2+ab}=\dfrac{b(a+b)}{a(a+b)}=\dfrac{b}{a}$.

26. AE=8, so AD=10 (DEA is a 3-4-5 right triangle scaled by a factor of 2). Hence the area of ABCD $=6\cdot 12=DF\cdot 10$, so $DF=7.2$

27. The distance from A to the foot of the perpendicular shown is $(16-8)/2=4$, so the triangle is a 3-4-5 right triangle and hence AB=5. So the perimeter is 8+16+2(5)=34.

28. $\dfrac{3\cdot BM}{2}=\dfrac{3\cdot 3}{2}$, so $BM=2$. Thus $CM=\sqrt{2^2+3^2}=\sqrt{13}$.

Hints and solutions for problems 29-44 will be added to these notes later.

45. C lies on the circle with the center at B and radius 4. If AD = 0 then A = D and hence C also lies on the circle with the center at A and radius 3. But these two circles do not intersect since AB = 10 > 4 + 3. Hence AD is larger than 0. On the other hand, if all four points lie on the same line then AD $=10-4-3=3$.

II Intermediate Problems (answers, hints, solutions)

I 1. Area of ABCD is 9; area of XYZW is 8.

I 2. Find a suitable pair of $36°-72°-72°$ triangles.

I 3. Let Q be the foot of the perpendicular from A to DP. Then DQA is a 6-8-10 right triangle. Since the sum of the angles QAD and ADQ is $90°$ and the sum of the angles PDC and ADQ is $90°$ as well, the angles QAD and PDC are equal to each other. Also, $\dfrac{AQ}{DP}=\dfrac{8}{12}=\dfrac{2}{3}$, and $\dfrac{AD}{DC}=\dfrac{10}{15}=\dfrac{2}{3}$. Thus the triangles QAP and PDC are similar and so $\angle DPC=\angle AQP=90°$.

I 4. Hint for parts (a) and (b): Area of the triangle ABC $=\dfrac{x\cdot AB}{2}+\dfrac{y\cdot AC}{2}+\dfrac{z\cdot BC}{2}=12$.

Hence $5(x+y+z)+z=24$, and so for any choice of P, $x+y+z=\dfrac{24-z}{5}$. Clearly, the larger is z, the smaller is the sum $x+y+z$, and vice versa. Try to investigate part (c) using the same approach.

I 5. $\Delta DAP \approx \Delta BAC \approx \Delta DBC \approx \Delta CBQ$: the first two right triangles are similar since they share an angle A; the second pair of triangles are similar since they both are right triangles and $\frac{BC}{AC} = \frac{12}{16} = \frac{3}{4} = \frac{9}{12} = \frac{DC}{BC}$; the last pair of right triangles share the angle DBC.

For part (b), comparing from similar triangles DAP and CBQ we get $\frac{QC}{PD} = \frac{BC}{AD} = \frac{12}{7}$.

For part (a), we get $\frac{Area\ of\ \Delta BCD}{Area\ of\ \Delta BAD} = \frac{(QC \cdot BD)/2}{(PD \cdot AB)/2} = \frac{QC}{PD} \cdot \frac{BD}{AB} = \frac{12}{7} \cdot \frac{DC}{BC} = \frac{12}{7} \cdot \frac{9}{12} = \frac{9}{7}$

($\frac{BD}{AB} = \frac{DC}{BC}$ since $\Delta DBC \approx \Delta BAC$.)

Of course, it's also possible to get part (a) first using scaling factors between similar triangles.

I 6. Let the side of the square be a. Then for the three shorter sides of the pentagon we have $a - x = x\sqrt{2} = a - x$, whence $x = \frac{a}{\sqrt{2}+1} = a(\sqrt{2} - 1)$. Therefore

$$\frac{Area\ of\ the\ pentagon}{Area\ of\ the\ square} = \frac{a^2 - \frac{x^2}{2}}{a^2} = \frac{a^2 - \frac{a^2(\sqrt{2}-1)^2}{2}}{a^2} = 1 - \frac{(\sqrt{2}-1)^2}{2} = 1 - (2 - 2\sqrt{2} + 1)/2 = \sqrt{2} - \frac{1}{2}$$

I 7. It is convenient to place the circle in the coordinate plane so that its center is at the origin O, and the square is inscribed in its part which is contained in the first quadrant.

If one of the vertices of the square is in the center than the other vertices are $(0, a), (a, a), (a, 0)$; since the point (a, a) lies on the circle $a^2 + a^2 = 10^2$. Hence the area of the square is $a^2 = \frac{10^2}{2} = 50$.

If none of the vertices of the square is at the center of the circle then obviously one vertex is on the x-axis, one vertex is on the y-axis, and two vertices are on the circle. Let the vertices on the coordinate axes be $B(0, b)$ and $A(a, 0)$, and let the square be $ABCD$. Let the horizontal line through C intersect the y-axis at P, and let the vertical line through D intersect the x-axis at Q. Clearly, $\Delta OBA \cong \Delta PCB \cong \Delta QAD$, and hence points C and D have coordinates $(b, a+b)$, and $(a+b, a)$, respectively. Thus $b^2 + (a+b)^2 = 10^2 = (a+b)^2 + a^2$, so that $b^2 = a^2$ and hence $a = b$. It follows that the coordinates of B are $(0, a)$, coordinates of C are $(a, 2a)$, and coordinates of D are $(2a, a)$, and we have $a^2 + 4a^2 = 10^2$. Hence

$$Area\ of\ the\ square = (side\ of\ the\ square)^2 = |OB|^2 + |OA|^2 = 2a^2 = \frac{2 \cdot 10^2}{5} = 40.$$

I 8. If the side is x then $x + .6 > 3.8$ and $x < 3.8 + .6$. Thus x is an integer between 3.2 and 4.4.

I 9. (a) Side of a square of the required area is the diagonal of a one by one square; (b) Side of a square of the desired area is the diagonal of a one by two rectangle.

I 10. Answer: ½

I 11. The answer is No. Hint: Consider their lids.

I 12. Answer: 7. Hint: Break every outer triangle into a pair of triangles of area 1 each.

Hints and solutions for Harder Problems will be added to these notes later.