Bad Squares on Board Games

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1 The Inspiration

Imagine a board game (such as *Monopoly*) in which the roll of two dice determines the number of squares you move forward on a given turn. A particularly "bad" square (TWO hotels on *Boardwalk*!) looms n squares ahead of your current square. How much should you panic? What is the probability that you land on the bad square?

2 Solving Simpler Problems First

A good way to begin thinking about a hard problem is to build your intuition and understanding by solving simpler problems and working out small examples. You can simplify the bad square problem, for example, by limiting yourself to one die.

Useful Fact: Let A and B denote possible outcomes of one or more rolls of a single die. For example, A might represent two rolls each showing 1 dot, followed by one roll showing 2 dots, and B could stand for one roll showing 4 dots. Then the

$$(\text{probability of } A \text{ and then } B) = (\text{probability of } A) \times (\text{probability of } B),$$

and if $A \neq B$, then the

$$(\text{probability of } A \text{ or } B) = (\text{probability of } A) + (\text{probability of } B).$$

Question 1: Let $P(n, m)$ denote the probability of landing n squares ahead in m rolls of a single die. What are the values of $P(n, m)$ for $1 \leq n, m \leq 6$?

Question 2: Let $P(n)$ denote the probability of landing n squares ahead in any number of rolls of a single die. Then $P(n) = P(n,1) + P(n,2) + \cdots + P(n,n)$. What are the values of $P(n)$ for $1 \leq n \leq 6$?

Question 3: Why can $P(n)$ can be written as $\frac{1}{6}\left(1+\frac{1}{6}\right)^{n-1}$ for $1 \leq n \leq 6$? (*Hint:* Think about how to expand $(x+y)^n$.)

Question 4: Does your formula for $P(n)$ work for $n > 6$? Why or why not?

3 The Problem

Question 5: Let q_n denote the probability of landing n squares ahead in a single roll of two dice. What are the values of q_n for $1 \leq n \leq 12$?

Question 6: Let $Q(n)$ denote the probability of landing n squares ahead in any number of rolls of two dice. What are the values of $Q(n)$ in terms of q_n for $1 \leq n \leq 12$?

Question 7: How do you compute $Q(n)$ for $n > 12$?

4 Further Insights

Let $C(n,k)$ denotes the number of ways to choose k different objects from a set of n distinct objects. Then

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

To make this formula work, define $0! = 1$.

Example: If you have the $n = 5$ distinct letters a, b, c, d, and e, and you want to choose $k = 3$ different ones, then there

$$C(5,3) = \frac{5!}{3!\,2!} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 2\cdot 1} = 10$$

different combinations: *a-b-c, a-b-d, a-b-e, a-c-d, a-c-e, a-d-e, b-c-d, b-c-e, b-d-e*, and *c-d-e*. (*Note:* the order in which the objects are chosen does not matter.)

The numbers $C(n,k)$ are called *binomial coefficients*. They are the numbers that appear in *Pascal's Triangle*. (Try computing a few!) The term "binomial" means "two names". The "names," for example, can be "x" and "y". When we expand $(x+y)^n$, the coefficients in the expansion are the numbers $C(n,0)$, $C(n,1)$, $C(n,2)$, …, and $C(n,n)$—those in the n^{th} row of *Pascal's Triangle*.

For example, $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$. The coefficients are $C(4,0) = 1$, $C(4,1) = 4$, $C(4,2) = 6$, $C(4,3) = 4$, and $C(4,4) = 1$.

Challenge Question 1: In answering questions 1 - 3, you may have observed that $P(n,m) = C(n-1, m-1)\left(\frac{1}{6}\right)^m$ when $1 \leq n, m \leq 6$. Is this just a coincidence, or is there a reason that computing $P(n,m)$ requires you to choose $m-1$ different objects from a set of $n-1$ distinct objects? What are the $n-1$ distinct objects? What do the $m-1$ that you choose represent?

Challenge Question 2: How can you use your answer to challenge question 1 to compute $P(n,m)$ for $n > 6$?

Challenge Question 3: Let $f(z) = \left(\frac{1}{6}z + \frac{1}{6}z^2 + \frac{1}{6}z^3 + \frac{1}{6}z^4 + \frac{1}{6}z^5 + \frac{1}{6}z^6\right)$. Why is $P(n)$ equal to the coefficient of z^n in the expansion of

$$f(z) + f(z)^2 + f(z)^3 + \ldots?$$

How does this help you?

5 Related Common Core Standards

Practice Standards

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

Content Standards

- Write and interpret numerical expressions. (5.OA)
- Analyze patters and relationships. (5.OA)
- Apply understanding of arithmetic to algebraic expressions. (6.EE)
- Represent and analyze quantitative relationships between dependent and independent quantities. (6.EE)
- Solve real-life and mathematical problems using numerical and algebraic expressions and equations. (7.EE)