$(\mathrm{CM})^2$ Intro Workshop: Divisibility Tests

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Let's write any integer a in terms of its decimal expansion: $a = a_0 + 10a_1 + 100a_2 + \cdots + a_d 10^d$ (so we're assuming a has $d+1$ digits). Our goal is to prove the following divisibility tests.

- a is divisible by 2 if and only if a_0 is.
- a is divisible by 4 if and only if $a_0 + 10a_1$ is.
- a is divisible by 8 if and only if $a_0 + 10a_1 + 100a_2$ is.
- Generalize the first three rules to divisibility by any power of 2.
- a is divisible by 5 if and only if a_0 is.
- a is divisible by 10 if and only if a_0 is.
- a is divisible by 3 if and only if $a_0 + a_1 + a_2 + \cdots$ is.
- a is divisible by 9 if and only if $a_0 + a_1 + a_2 + \cdots$ is.
- a is divisible by 11 if and only if $a_0 - a_1 + a_2 - \cdots$ is.
- a is divisible by 7 if and only if $(a_0 + 10a_1 + 100a_2) - (a_3 + 10a_4 + 100a_5) + (a_6 + 10a_7 + 100a_8) - \cdots$ is.
- a is divisible by 11 if and only if $(a_0 + 10a_1 + 100a_2) - (a_3 + 10a_4 + 100a_5) + (a_6 + 10a_7 + 100a_8) - \cdots$ is.
- a is divisible by 13 if and only if $(a_0 + 10a_1 + 100a_2) - (a_3 + 10a_4 + 100a_5) + (a_6 + 10a_7 + 100a_8) - \cdots$ is.
- $a = 10x + y$ is divisible by 7 if and only if $x - 2y$ is.
- a is divisible by 7 if and only if $(a_0 + 3a_1 + 2a_2) - (a_3 + 3a_4 + 2a_5) + (a_6 + 3a_7 + 2a_8) - \cdots$ is.
- a is divisible by 7 if and only if $(-2)^d a_0 + (-2)^{d-1} a_1 + (-2)^{d-2} a_2 + \cdots + a_d$ is.[1]
- a is divisible by 17 if and only if $(-5)^d a_0 + (-5)^{d-1} a_1 + (-5)^{d-2} a_2 + \cdots + a_d$ is.
- a is divisible by 19 if and only if $2^d a_0 + 2^{d-1} a_1 + 2^{d-2} a_2 + \cdots + a_d$ is.

[1] This and the following two tests we learned from Apoorva Khare, a sixth form student in Orissa, India, see Electronic Journal of Undergraduate Mathematics **3** (1997), 1–5.

The easiest way to explain these rules is through modular arithmetic. Some of the rules above have counterparts in the language of modular arithmetic, which are stronger statements:

- $a \equiv a_0 \pmod 2$
- $a \equiv a_0 + 10a_1 \pmod 4$
- $a \equiv a_0 + 10a_1 + 100a_2 \pmod 8$
- $a \equiv a_0 \pmod 5$
- $a \equiv a_0 \pmod{10}$
- $a \equiv a_0 + a_1 + a_2 + \cdots \pmod 3$
- $a \equiv a_0 + a_1 + a_2 + \cdots \pmod 9$
- $a \equiv a_0 - a_1 + a_2 - \cdots \pmod{11}$
- $a \equiv (a_0 + 10a_1 + 100a_2) - (a_3 + 10a_4 + 100a_5) + (a_6 + 10a_7 + 100a_8) - \cdots \pmod 7$
- $a \equiv (a_0 + 10a_1 + 100a_2) - (a_3 + 10a_4 + 100a_5) + (a_6 + 10a_7 + 100a_8) - \cdots \pmod{11}$
- $a \equiv (a_0 + 10a_1 + 100a_2) - (a_3 + 10a_4 + 100a_5) + (a_6 + 10a_7 + 100a_8) - \cdots \pmod{13}$.
- $a \equiv (a_0 + 3a_1 + 2a_2) - (a_3 + 3a_4 + 2a_5) + (a_6 + 3a_7 + 2a_8) - \cdots \pmod 7$.