Rubik's Slide

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version for leaders

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Abstract algebra functions	Using smaller problems Organizing material Notation

1 Defining the Game

Setup

We're going to play a variation of Rubik's cube. This is actually an electronic game but we're just going to do it on paper - you can imagine it with LED lights, etc, to make it more exciting!

Game Rules There is flat display of a 3×3 grid. The goal of the game is for the player to transform a given initial state into a given final state using a set of allowed moves. By a "state" we mean that each square in the grid is colored a certain color or left blank. In the actual electronic game the player is shown the initial state on the grid and by pushing a button, the player can see what the final state should be. We'll just draw both on paper here.

Allowed Moves There are 3 types of moves, described below in words and with a picture (which labels our grid with numbers 1 through 9)

I) shift all columns to the right. Here is what it looks like:

1	**2**	**3**
4	**5**	**6**
7	**8**	**9**

becomes

3	**1**	**2**
6	**4**	**5**
9	**7**	**8**

One can similarly shift to the left (i.e. go backwards on the above).

II) shift all rows down. Here is what it looks like:

1	**2**	**3**		**7**	**8**	**9**
4	**5**	**6**	becomes	**1**	**2**	**3**
7	**8**	**9**		**4**	**5**	**6**

One can similarly shift the rows up (i.e. go backwards on the above).

III) shift clockwise. Here is what it looks like:

1	**2**	**3**		**4**	**1**	**2**
4	**5**	**6**	becomes	**7**	**5**	**3**
7	**8**	**9**		**8**	**9**	**6**

One can similarly shift counterclockwise (i.e. go backwards on this).

2 Some Practice Games

When playing the electronic game, the initial and final states are shown using color; here we'll do by using the letter X. Go ahead and try to figure out how to get from the initial state to the final state, using only the allowed rules.

2.1 First Practice Game

Initial state:

		X

Final state

	X	

2.2 Second Practice Game

Initial state:

X	**X**	

Final state

X		
X		

3 Main Question

Given any configuration of X's and any other configuration of the same number of X's, can one use these allowed moves to get from one to another?

4 Hints and comments to make

4.1 Notation

First off, note that if we are going to compare answers, we will need to agree on some common notation (i.e. common language). Let's use

h for a horizontal shift to the right (and thus h^{-1} for the horizontal shift to the left),

v for a vertical shift down (so v^{-1} for the vertical shift up), and

c for a clockwise shift (so c^{-1} for counterclockwise).

[Also note that these are functions]

4.2 Smaller Problem

Try a smaller problem! Given two configurations, each with one X, can you get from one to the other?

[They might start by looking if the beginning X and ending X are in the same row or column, then if one is in middle]

[Make sure they can articulate their process!]

Given two configurations, each with two X's, can one get from one to the other?

Note that it is enough here to be able to get from an arbitrary configuration to any fixed one, as then we can go backwards between two.

[They may start by looking if the two X's are in the same row/column, then if they are diagonal to one another]

4.3 Cases

One can break these into cases: when X's lie on same line, or when not. Etc.

4.4 Answers to Examples

There is more than one way to do each problem, but possibilities are:

First Game: $v^{-2}h^{-1}$, i.e. shift the columns to the left and then shift the rows up twice.

Second Game: $v^{-1}\,h^{-1}\,c^2\,v^{-1}$, i.e. shift the rows up, then rotate clockwise twice. This lines the X's in squares 2 and 5. Shift the columns to the left and then up.

5 Answer to Main Question

The answer is yes, and this follows from first noting that you can interchange two locations and leave all other locations fixed by using $(h\,c^3)^7$, i.e. rotate clockwise 3 times and then shift to the right once: repeat 7 times. Now break any problem into successive steps, each of which involves interchanging two positions. More explicitly, this follows from the fact that a cycle $(a\,b\,c) = (c\,b)\,(c\,a)$, etc.

6 Other questions

6.1 Hard Mode

If one plays Rubik's Slide in "hard mode", it can have multiple colors. Can we still get from any configuration to any other configuration (with the same number of colors)?

6.2 Answer

Yes, for the same reason as above: you can switch any two locations without affecting the others, then do that successively.

6.3 Easy Mode

If one plays Rubik's Slide in "easy mode", the allowed moves are slightly different. There is still h and v (and their inverses) but instead of c, there is only c^2. Now is it always possible to solve a puzzle?

6.4 Answer

No!

If there is only one colored square then any other configuration with one colored square can be solved. But if there are two different colored squares then it is not always possible, for instance consider

X	**Y**	

and

X		
	Y	

Neither can be changed by our allowed moves into the other: note that with our new allowed moves, two squares that share an edge will still share on edge after an allowed move (thinking of the top and bottom as sharing an edge, and same with the right and left). But X and Y share an edge in the first and don't in the second.

6.5 Given an initial state, how many final states can appear that can be solved in the easy mode?

6.6 Answer

If there is only one colored square then any other configuration with one square can be put in, as one can use a combination of $\{h^{\pm 1}, v^{\pm 1}\}$ to solve. If there are two colored squares, each a different color, then there are two distinct families, each with 36 members (72/2). If there are two colored squares with only one color, then family is of size 18.

7 Wrap Up

7.1 Connections

7.1.1 Things you've seen

Note that the allowed moves are actually functions, which have the numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as the domain and the range. In function notation we might write $h(x)$ as the "shift to the right" function, so that $h(1) = 2$, $h(2) = 3$, $h(3) = 1$, etc.

We said we could go backwards on these, which gives us the inverse function, written h^{-1}. Thus $h^{-1}(2) = 1$, $h^{-1}(3) = 2$, etc.

7.1.2 Things you may not have seen

The functions above are called permutations (of 9 elements), and we could consider the collection of all such things. This collection is called the symmetric group on 9 elements, denoted S_9. This is an example of a group, which is one of the basic objects studied in Abstract Algebra. In Rubik's Slide, you are given 3 elements from S_9 and asked if you can use these to get all the other elements in S_9, i.e. do these 3 elements generate the group?

Another example of a group you may have seen is when you look at all symmetries of a polygon (often involves rotation and flips)

7.2 What Problem Solving Techniques Did You Use?

Using cases, using notation, going backwards, doing an easier problem.

7.3 Thinking like a Mathematician

What other questions might you be interested in about this scenario?

What is the minimum number of moves needed, do we really need all 3 moves (no).

7.4 Discourse

What questions helped you to understand more?

Asking "Why is that true", etc.

7.5 References

Check out related information on our webpage, philamtc.blogspot.com/

Also, we noticed information about Rubik's Slide on this mathblog: http://mathchat.wordpress.com/2011/05/01/rubiks-slide-play-your-way-to-geometric-knowledge

Finally, the idea for this project came from an article in the American Mathematical Monthly: *Rubik's on the Torus* by Jeremy Alm, Michael Gramelspacher, and Theodore Rice, February 2013.