MAGIC SQUARES

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Supplies: Paper and pen(cil)

1. Initial setup

Today's topic is *magic squares*. We'll start with two examples. The unique magic square of order one is $\boxed{1}$.

An example of a magic square of order five is:

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

There are a few things to notice about these examples: the magic square of order one is a square 1-by-1 array that contains only the number 1; the magic square of order five is a 5-by-5 square array containing the numbers $1, 2, 3, \ldots, 25$. Each number in the list appears exactly once. Further, if you take the sum of any row ($17 + 24 + 1 + 8 + 15$, for example), or any column ($1+7+13+19+25$, for example), it will always be the same number, called the *magic sum* (in this case, 65). In fact, in order for an array to be a magic square, we also require that the two main diagonals also sum to that same number. This brings us to the general definition.

A (normal) *magic square of order* n is an n-by-n square arrangement of the numbers $\{1, \ldots, n^2\}$ where:

- each number appears exactly once;
- every row, column, and the two main diagonals sum to the same number, the *magic sum*.

The first recorded examples of magic squares showed up over 2000 years ago, and they were long thought to have mystical properties, as their name suggests.

Squares that contain entries from any set of n^2 positive integers, instead of just $\{1, \ldots, n^2\}$ are also known as magic squares. The ones that specifically use the numbers $\{1, \ldots, n^2\}$ are sometimes called *normal magic squares.*

Date: February 22, 2015.

This session was prepared with the cooperation of the Philadelphia Area Math Teachers' Circle Leadership Team: Catherine Anderson, Kathy Boyle, Aimee Johnson, Amy Myers, Josh Sabloff, and Josh Taton.

Unless otherwise stated, we'll be working with normal magic squares (and so we'll drop the "normal" most of the time).

Question 1. Group discussion: Is there a magic square of order 2?

2. Exploration

Question 2. Construct a magic square of order 3.

Question 3. For a magic square of order 3 (containing the numbers $\{1, 2, \ldots, 9\}$), what does the magic sum *have* to be? Try to use a general argument that does not rely on the constructed squares.

Question 4. How many distinct magic squares of order 3 are there? When do we consider two squares to be the same?

Question 5. What is the magic sum of a magic square of order n?

3. Additional Questions

Notice that the middle entry of every order 3 magic square is 5, and the middle entry of the order 5 magic square example above is 13. Does the middle entry of an odd-order magic square always have to be the median number in the list $\{1, \ldots, n^2\}$?

4. Variations on Magic Squares

There are many variations of magic squares, for example: using multiplication instead of addition; magic triangles, cubes, or circles; magic squares within a larger square, similar to Sudoku; concentric magic squares; semi-magic squares that do not satisfy the diagonal sum condition, and many others!

Question 6. Choose one of the variations above, or make up your own. Can you construct one of these magic objects?

5. Sources

- W. Benson and O. Jacoby, "New recreations with magic squares," Dover Publications, Inc. (1976).
- P. Pasles, "Benjamin Franklin's Numbers: An Unsung Mathematical Odyssey," Princeton University Press (2008).
- E. W. Weisstein, "Magic Square," From *MathWorld–A Wolfram Web Resource*. `http://mathworld.wolfram.com/MagicSquare.html`, accessed February 10, 2014.
- D. Hawley, "Magic Squares," `http://nrich.maths.org/1337`, accessed February 10, 2014.