Using the Euclidean Algorithm

The Decanting Problem is a liquid measuring problem that begins with two unmarked decanters with capacities a and b. Usually a and b are integers. The problem is to determine the smallest amount of liquid that can be measured and how such amount can be obtained, by a process of filling, pouring, and dumping. Specifically, there are three actions we can take:

1. fill an empty decanter,
2. dump out a full decanter, and
3. pour from one decanter to the other until either the receiving decanter is full or the poured decanter is empty.

Let's look at an easy one first. Let $a = 3$ and $b = 5$. We can fill the 3 unit decanter twice, and dump the 5 unit decanter once to get 1 unit of liquid. Algebraically, $2 \cdot 3 - 1 \cdot 5 = 1$. Next, suppose the decanters have capacities 5 units and 7 units. A little experimentation leads to the conclusion that 1 unit of water can be obtained by filling the 5 unit decanter 3 times, pouring repeatedly from the 5 unit to the 7 unit decanter and dumping out the 7 unit decanter twice. A finite state diagram is helpful to follow the procedure:

$$(0,0) \Longrightarrow (5,0) \Longrightarrow (0,5) \Longrightarrow (5,5) \Longrightarrow (3,7) \Longrightarrow (3,0)$$

$$\Longrightarrow (0,3) \Longrightarrow (5,3) \Longrightarrow (1,7) \Longrightarrow (1,0),$$

where the notation (x, y) means the 5-unit container has x units of liquid and the 7-unit container has y units. Notice that the procedure includes 3 fills and 2 dumps, with fills and dumps alternating and separated by 4 pours. An arithmetic equation representing this is

$$3 \cdot 5 - 2 \cdot 7 = 1.$$

Notice that not only does the arithmetic equation follow from the state diagram, the reverse is also true. That is, given the arithmetic equation, it is an easy matter to construct the state diagram.

In the next example, the least amount that can be measured is not 1. Let the decanters have sizes 15 and 99. Before reading on, can you see why it is impossible to obtain exactly one unit of water? An equation can be obtained for any sequence of moves. Such an equation is of the form

$$15x + 99y = z$$

where exactly one of the integers x and y is negative, and z is the amount obtained. Now notice that the left side is a multiple of 3, so the right side must be also. Thus the least positive amount that can be measured is a multiple of 3. One can also reason this as follows: each fill adds a multiple of 3 units of water to the total amount on hand, each pour leaves the total number unchanged, and each dump removes a multiple of three units from the total, so the amount on hand at each stage is a multiple of 3.

In general, when a and b are integers, the least amount that can be measured is the greatest common divisor of the two decanter sizes, and the Euclidean algorithm, as explained below, tells us how to proceed. Suppose $c = GCD(a, b)$. The Euclidean algorithm yields a solution to

$$c = ax + by$$

where x and y are integers exactly one of which is positive and, except in trivial cases, the other is negative. For convenience, we assume x is positive. Then the solution to the decanting problem is to fill the a capacity decanter x times, repeatedly pouring its contents into the b unit decanter. The b unit decanter will be dumped y times, so the total liquid on hand at the end is the difference $ax - by = c$.

Let's look at another specific example. Again we use the Euclidean Algorithm to solve the decanting problem. There are two stages. The first stage is a sequence of divisions. The second is a sequence of 'unwindings'. For this example, let the decanter sizes be $a = 257$ and $b = 341$. Use the division algorithm to get $341 = 1 \cdot 257 + 84$. Then divide 257 by 84 to get $q = 3$ and $r = 5$. That is, $257 = 3 \cdot 84 + 5$. Continue dividing until the dividend is less than the divisor. Thus 84 divided by 5 yields $84 = 16 \cdot 5 + 4$. Finally, divide 5 by 4 to get $5 = 1 \cdot 4 + 1$. This completes the first stage. Now to unwind, start with the final division scheme, writing $1 = 5 - 1 \cdot 4$. Then replace the 4 with $84 - 16 \cdot 5$ to get $1 = 5 - 1(84 - 16 \cdot 5)$. This is equivalent to $1 = 17 \cdot 5 - 1 \cdot 84$. Check this to be sure. Then replace 5 with $257 - 3 \cdot 84$ to get

$$1 = 17 \cdot (257 - 3 \cdot 84) - 1 \cdot 84,$$

i.e., $1 = 17 \cdot 257 - 52 \cdot 84$. Finally, replace 84 with $341 - 257$ to get $1 = 17 \cdot 257 - 52(341 - 257)$, which we can rewrite as

$$1 = 69 \cdot 257 - 52 \cdot 341.$$

Thus, the solution to the decanting problem is to measure out 1 unit of liquid by filling the 257 unit decanter 69 times, repeatedly pouring its contents into the 341 unit decanter, and, in the process, dumping out the 341 unit decanter 52 times.

There is a related problem sometimes called the postage stamp problem. Here we are given an unlimited supply of two denominations of postage, a and b. If a and b are not relatively prime, and say $d > 1$ is the gcd of a and b, then there is no hope of specifying n cents in postage using as and bs unless n is a multiple of d. If a and b are relatively prime, then there is a largest number k that cannot be made. Here we are repeatedly solving the problem $n = ax + by, x, y \geq 0$.

Theorem Suppose m and n are relatively prime positive integers such that $m > n \geq 2$. Then

1. The equation $mx + ny = mn - m - n$ has no solution with $x \geq 0$ and $y \geq 0$.
2. For all $t \geq 1$, the equation $mx + ny = mn - m - n + t$ has a solution with $x \geq 0$ and $y \geq 0$.

Proof

1. Suppose that $mx+ny = mn-m-n$ with $x \geq 0$ and $y \geq 0$. Since $mn-m-n$ is a positive number that is not a multiple of either m or n, both x and y are positive. Next note that $xm = mn - m - n - yn = mn - m - (y + 1)n$ which implies that $m \mid (y + 1)n$. But since m and n are relatively prime, it follows that $m \mid y + 1$. Therefore $y + 1 = cm$ for some $c \geq 1$. It follows that $y = cm - 1$. Likewise there is a $d \geq 1$ such that $x = dn - 1$. Putting all this together, we have $(dn - 1)m + (cm - 1)n = mn - m - n$, and from this it follows that $(c + d)mn = mn$. But this can happen only when $c + d = 1$, a contradiction.

2. We show that for all $t \geq 1$, there exists $x \geq 0$ and $y \geq 0$ such that $mx+ny = mn-m-n+t$. By the Euclidean algorithm, there exists integers $\overline{x}, \overline{y}$ satisfying $m\overline{x}+n\overline{y} = mn-m-n+t$. Therefore, , for all integers c, $x = \overline{x}+cn, y = \overline{y}-cn$ also satisfy $mx + ny = mn - m - n + t$. Since $n < m$ we can choose c so that $0 \leq x \leq n-1$, and $mx+ny = mn-m-n+t$. If $y \geq 0$, we are done. Suppose $y \leq -1$. Then $mx + ny \leq (n-1)m - n = mn - m - n$ which contradicts the fact that $mx + ny = mn - m - n + t$, where $t \geq 1$. This complete the proof of the theorem.

Problems

For each pair of values s and t below, use repeated division to find $gcd(s,t)$ the greatest common divisor of s and t and then use the Euclidean Algorithm to solve the equation $gcd(s,t) = xs + yt$, where x and y are integers. In other words, solve the decanting problems for decanters of sizes s and t.

1. $s = 22$ and $t = 37$
2. $s = 105$ and $t = 95$
3. $s = 89$ and $t = 144$
4. Suppose you have decanters of sizes 99 and 105. Find the least amount of liquid that can be measured, show how to measure that amount, and explain in English why you cannot do better.
5. **Dinner Bill Splitting.** Years ago, my neighbors agreed to celebrate our wedding anniversary with my wife and me. The four of us went to a lovely restaurant, enjoyed a fine dinner, and asked for the bill. When it came, we asked that it be split in half. Realizing the waiter's discomfort, we all set to work on the problem. The bill was for an odd amount, so it could not be split perfectly. However, we realized that, except for the penny problem, we could take half the bill by simply reversing the dollars and the cents. In other words, if we double t dollars and s cents, the result differs by 1 cent from s dollars and t cents. We told the waiter about this. He was astounded: "I never knew you could do it that way." Later, over another dinner with mathematical friends, the question of uniqueness came up, and pretty soon we realized that this number is the only one with this surprising splitting property. What was the amount of the original bill?
6. Chicken McNuggets can be purchased in quantities of 6, 9, and 20 pieces. You can buy exactly 15 pieces by purchasing a 6 and a 9, but you can't buy exactly 10 McNuggets. What is the largest number of McNuggets that can NOT be purchased?

   See `http://www.mathnerds.com/mathnerds/best/mcnuggets/solution.aspx`
7. Does the equation $399x + 703y = 114$ have an integer solution in x and y?
8. Does the equation $399x + 703y = 115$ have an integer solution in x and y?

9. For m and n integers, characterize those integers k for which the equation $mx + ny = k$ has integer solutions in x and y.

10. **The Subtraction Game** In the Subtraction Game, two players start with some positive integers written on a board. The first player must find a pair of numbers whose positive difference is not already written on the board. Then he writes this new number on the board. At each stage, the next player finds a positive difference between two numbers on the board that is not already written on the board and writes it on the board. The first player who cannot find a new positive difference loses. For each of the sets of numbers listed below, decide how many numbers will be on the board at the end of the game. Use this information to state whether you would like to move first or not (in order to win).

    (a) $101, 102, 103$

    (b) 3105 and 4104

    (c) $21, 24, 81, 87$