

### Problem Solving Warm-ups

1. Find a 9-digit number using each digit 1 through 9 once, such that the number formed by the first  $n$  digits is divisible by  $n$ .
2. What's the smallest number that can't be written by subtracting a prime from a square? For example,  $1 = 4 - 3$ ,  $2 = 9 - 7 \dots$
3. Around the Round Table sit the 64 knights. Sir Cumference then walks around and around the table, saying alternately go, stay, go, stay for each knight he passes. Those told "go" get up and leave. If this process is continued until only one knight is left sitting, which knight will be the last one remaining? Can you generalize?
4. Use the four integers 94, 95, 96, 97 to make two fractions (so two of these numbers are in the numerator and two in the denominator.) Which arrangement gives the largest possible sum of these two fractions? The smallest? For example, one such sum is  $\frac{95}{97} + \frac{96}{94}$ .
5. Add up all the numbers in the usual multiplication table shown below. What are different ways to find that sum?

	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10</b>	<b>11</b>	<b>12</b>
<b>1</b>	1	2	3	4	5	6	7	8	9	10	11	12
<b>2</b>	2	4	6	8	10	12	14	16	18	20	22	24
<b>3</b>	3	6	9	12	15	18	21	24	27	30	33	36
<b>4</b>	4	8	12	16	20	24	28	32	36	40	44	48
<b>5</b>	5	10	15	20	25	30	35	40	45	50	55	60
<b>6</b>	6	12	18	24	30	36	42	48	54	60	66	72
<b>7</b>	7	14	21	28	35	42	49	56	63	70	77	84
<b>8</b>	8	16	24	32	40	48	56	64	72	80	88	96
<b>9</b>	9	18	27	36	45	54	63	72	81	90	99	108
<b>10</b>	10	20	30	40	50	60	70	80	90	100	110	120
<b>11</b>	11	22	33	44	55	66	77	88	99	110	121	132
<b>12</b>	12	24	36	48	60	72	84	96	108	120	132	144

6. How many rectangles are there in a 13x13 chessboard?

7. Let  $n$  be an odd integer and write the integers  $1, 2, \dots, 2n$  on a board. Allie picks two of the integers, say  $a$  and  $b$ , erases them then writes the integer  $|a - b|$  on the board. This process is repeated until there is just one number left. Could this number be even?
8. A cube begins with one vertex labeled 1 and all the rest labeled 0. At any moment, you can add 1 to each of any two adjacent vertices. Can you ever make the vertices all have multiples of 3?
9. On a boring rainy day you play the following solitaire game in your garage. You start with a stack of 20 boxes. For any turn, you select a pile, make it into two piles, then take the product of the numbers of boxes in each pile and add to your score. This process continues until you have 20 piles of one box each. What is the highest score you can get?
10. Here is a mathemagic trick you can use to amaze your students. Turn your back to the blackboard and have a student pick two positive integers and write them one below the other on the board. (You might suggest two digit numbers because there is a lot of addition to be done after this!). You of course cannot see the numbers nor are you told what they are. Below this have the student write the sum of the two numbers, so there is now a column of 3 numbers on the board. Then add the bottom two numbers in the column and put the result of this addition below the three. Continue in this way, adding the two most recent numbers in the column and writing the sum below, until there are 10 numbers in the column. You then declare that you will turn and almost instantly find the sum of the ten numbers. For example, if the starting numbers were 23 and 89, we would end up with the following list:

23  
 89  
 112  
 201  
 313  
 514  
 827  
 1341  
 2168  
 3509

and you could immediately declare that the sum is 9097. How does this trick work? (Caution: This will not work if the student makes an addition mistake, so pick one of your best students!)

## Problem Solving Warm-ups—Notes

1. *Technique: Patience and educated guessing.* Most of the work is trial and adjustment, along with some things you know must be true given the requirements. For example, the even digits must be in places 2, 4, 6, 8; the fifth digit must be 5; the last digit does not matter because the sum of the digits is divisible by 9 and this means every number using these 9 digits is divisible by 9; etc.
2. *Technique: Try small cases, try to infer a general principle.* It is not hard to find that 16 cannot be so written. There is more going on here though because there can be problems with other squares. If  $N = n^2$  is square and  $N = n^2 = m^2 - p$  is a difference of the square  $m^2$  and the prime  $p$ , then  $p = m^2 - n^2 = (m - n)(m + n)$ . Because  $p$  is prime we must have  $m - n = 1$  so  $p = m + n = 2n + 1$ . Thus if  $2n + 1$  is not prime, this representation is impossible. For the case  $N = 16 = 4^2$ ,  $2n + 1 = 2 \cdot 4 + 1 = 9$  is not prime!
3. *Technique: Try small cases, relate the given case to the smaller cases.* Actually there are many ways to do this problem. It can be done by brute force (e.g. just do it!). Also, after one complete turn around the table note that the problem is reduced to the 32 knight case (Why?). A third way is to write the knight numbers in base two. With each round what is special about the base-two representations of the numbers eliminated?
4. *Technique: Patience and educated guessing.* It is certainly not hard to try all possibilities. However several of the possibilities can be eliminated by common sense and the few remaining can be checked by hand (or calculator.)
5. *Technique: Organization.* There are many ways to do this and some ways are more efficient than others if things are organized properly. Probably the quickest way to do this one is to think about the distributive property:

$$\text{sum} = (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12)^2.$$

6. *Technique: Organization.* Think of some way of grouping rectangles into categories and then counting the number of rectangles in each category. When you observe that this problem has the same answer as the previous problem, go back and see if you can see a connection between the two.

7. *Find an invariant.* An invariant is something that does not change. Invariants can be very useful in analyzing problems that start in one state and ask if you can get to another state by repeating a certain operation. Invariants can be hard to find but when found can lead to very beautiful solutions. Parity (odd/even) is a very common invariant. In this problem, what can you say about the sum of all of the integers on the board after each operation?
8. Assign to each vertex of the cube a  $+$  sign or a  $-$  sign, assigning 4 of each in such a way that each edge of the cube has different signs at its two ends. Calculate the sum of the numbers at the vertices with  $+$  signs and subtract the sum from the vertices with  $-$  signs. What is the result?
9. *Try simple cases, guess a general principle, prove it.* Try this problem with smaller stacks of boxes. With enough data you can predict the the answer to the 20 box problem. Why is this problem just the handshake problem in disguise? (e.g., 20 people are in a room, each shakes hands with each of the others. How many handshakes total?) So another very useful technique is to notice when a problem can be recast as a more familiar problem.
10. *Generalize.* Sometimes it is easier to see what is happening with a general case rather than a specific case. Rather than starting with the number 23 and 89, do the problem and find the sum starting with the numbers  $a$  and  $b$ . What do you notice about the seventh number in the list and the sum of the 10 numbers?