Handshakes and Graphs and Ramsey Theory

Warm-up problems.

a. Twenty people are in a room. If each of the people shakes hands exactly once with each of the other people, what is the total number of handshakes?

b. At the start of an mathematics conference between the eastern NCTM and the western NCTM, each teacher shook hands with every other member of his/her own group for a total of 466 handshakes. Next, each teacher shook hands with every person in the *other* group for 480 more handshakes. What was the total number of teachers at the conference?

Graph Theory is a very popular and powerful way to investigate problems such as these (and the ones below.) A *graph* consists of a finite collection of points (vertices) and arcs or segments (edges) connecting some pairs of vertices. It is OK if, in drawing the graph, you have to make two edges cross…just keep in mind that there is no vertex at such points. (In fact, the study of which graphs can be drawn without having edges cross is a an active area of research in graph theory.)

The situations in the warm-up problems can be represented by graphs: represent each person as a vertex, and connect a pair of vertices by an edge if and only if the corresponding pair of people shook hands. In the warm-up problem a., there is an edge between every pair of vertices because every pair of people shake hands. The resulting graph is called *the complete graph on 20 vertices.*

Graph Theory is a very important tool in many aspect of modern life. For example, it shows up in airline scheduling, communications, computer programming, circuit design, the analysis of chicken pecking orders, etc.

1. Six people are in a room. Some of the pairs shake hands. Prove that either there is a set of three people each of whom have shaken hands with the other two, or a set of three people each of whom has not shaken hands with either of the other two.

2. You are given a complete graph on six vertices. You color each edge either red or blue (every edge is colored!) Prove that there will always be a triangle all of whose sides are the same color.

3. A room has 9 people. Some pairs shake hands and some do not. Is it possible that each person shakes hands exactly 3 times?

4. Again there are 20 people in a room. Suppose some pairs of the people shake hands and some don't. As the people leave the room you (who were not in the room) ask each person whether they shook hands and odd number of times or an even number of times. Prove that the number of people who answer "odd" is an even number.

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5. A man and wife invite 5 other couples to a party, for a total of 12 people. During the party, some pairs of people shake hands, but nobody shakes his or her spouse's hand. Later in the party, the host asks each of the others how many other people they have shaken hands with. As it turns out, he gets a different answer from each person. With how many people did the host shake hands?
6. Consider the complete graph on 17 vertices. Each edge is colored either red or blue or green. Prove that there must be a triangle with all sides the same color. (Hint: Problems 1 and 2 can be very helpful here!)
7. Again you have a room of people and some pairs of the people shake hands and some don't. You want to find a group of four people, each of whom has shaken hands with the other three, or a group of four, for which no pair has shaken hands. How many people must be in the room to guarantee that no matter how the hand shaking goes, such a group of four can be found? (Don't try to solve this one, but make a guess as to how many people you would need.)